UNIT: II TIME RESPONSE ANALYSIS :

Transient & steady state response - Measures of performance of first order & second order system - Effect of additional pole & additional zero - steady state error constant & system - Type number - PID controller - Analytical design of PI, PD, PID controller.

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Introduction:

Time response analysis is also called time domain analysis. ie., output as a function of time.

Total time response $c(t)$ of a control system consists of transient response (dynamic response) $c_t(t)$ and steady state response $c_{ss}(t)$.

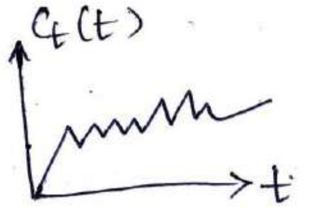
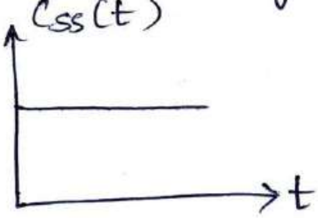
$$\boxed{c(t) = c_t(t) + c_{ss}(t)}$$

$c_{ss}(t)$ → steady state response

$c_t(t)$ → Transient response

$c(t)$ → Total time response.

A feedback control system has the inherent capabilities that its parameters can be adjusted to alter both its transient and steady-state behaviour.

Transient Response	Steady state Response
It remains for a Very short time.	It remains as time 't' approaches infinity (longtime)
	
It depends on system poles only. not on type of input.	It depends on both system poles and type of input.

Before proceeding with time response analysis of a control system, it is necessary to test stability of the system through indirect tests without actually obtaining the transient response. In case, system is unstable. hence of no practical use.. we need not proceed with its transient response Analysis.

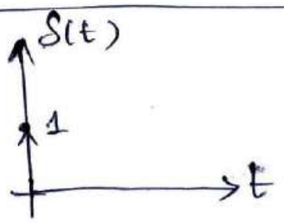
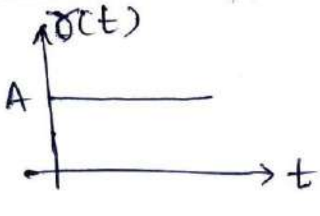
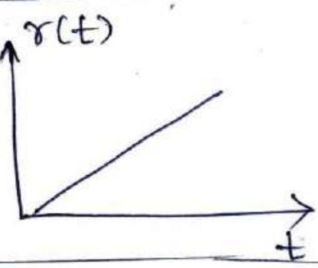
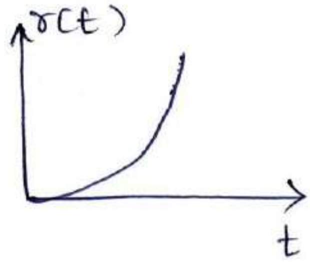
Typically test signals generated in laboratory are,

(i) unit Impulse (sudden shock)

(ii) unit step (sudden change)

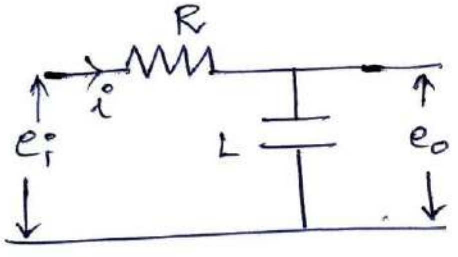
(iii) Ramp (constant Velocity)

(iv) parabolic (constant Acceleration).

Sno	Signal	Diagram	Input $r(t)$	output $R(s)$
1	unit Impulse		$r(t)=\delta(t) = \begin{cases} 1, & t=0 \\ 0, & t\neq 0 \end{cases}$	$R(s) = 1$
2.	unit step		$r(t) = u(t) = \begin{cases} A, & t>0 \\ 0, & t<0 \end{cases}$	$R(s) = \frac{1}{s}$
3.	Ramp		$r(t) = \begin{cases} t, & t>0 \\ 0, & t<0 \end{cases}$	$R(s) = \frac{1}{s^2}$
4.	parabolic signal		$r(t) = \begin{cases} \frac{t^2}{2}, & t\geq 0 \\ 0, & o.w \end{cases}$	$R(s) = 1/s^3$

Measures of performance of First order system.

Let us consider a simple RC circuit,.



RC circuit of first order s/m.

$$e_i = Ri + \frac{1}{C}\int i\,dt \longrightarrow (1)$$

$$e_o = \frac{1}{C}\int i\,dt \longrightarrow (2)$$

Taking Laplace transform for (1) & (2)

$$E_i(s) = \left[R + \frac{1}{Cs}\right] I(s) \longrightarrow (3)$$

$$E_o(s) = \frac{1}{Cs} I(s) \longrightarrow (4)$$

Transfer function: $\dfrac{E_o(s)}{E_i(s)} = \dfrac{(1/Cs)\,\cancel{I(s)}}{(R + 1/Cs)\,\cancel{I(s)}}$

$$\frac{E_o(s)}{E_i(s)} = \frac{\cancel{Cs}}{(1+RCs)\,\cancel{Cs}} = \frac{1}{1+RCs} \longrightarrow (5)$$

where $\tau = RC$ = Time constant.

$$\therefore \boxed{T.F = \frac{1}{1+\tau s}}$$

$$\Rightarrow T.F = \frac{C(s)}{R(s)} = \frac{1}{1+\tau s}$$

$$\boxed{C(s) = R(s)\left(\frac{1}{1+\tau s}\right)} \longrightarrow \text{Main Equation.}$$

<u>Input: unit impulse s/g</u>

$$R(s) = 1$$

$$\therefore C(s) = \frac{1}{1+\tau s} \longrightarrow (6) \qquad C(s) = \frac{1}{\tau(s+1/\tau)}$$

Taking inverse L.T $\Rightarrow$ $\boxed{c(t) = \frac{1}{\tau} e^{-t/\tau}} \longrightarrow (7)$

steady state error:-

$$e_{ss} = \lim_{t\to\infty} [e(t)] = \lim_{t\to\infty} [r(t) - c(t)]$$

[or]

$$e_{ss} = \lim_{s\to 0} s[E(s)] = \lim_{s\to 0} s[R(s) - C(s)] \rightarrow 8$$

Substitute R(s) & C(s) value in 8.

$$\therefore e_{ss} = \lim_{s\to 0} s\left[1 - \frac{1}{\tau s+1}\right]$$

$$\boxed{e_{ss} = 0}$$

Input: unit step s/g

$R(s) = 1/s$ substitute in main eqn

$$\therefore C(s) = \left(\frac{1}{s}\right)\left(\frac{1}{1+\tau s}\right)$$

$$\frac{1}{s(1+\tau s)} = \frac{A}{s} + \frac{B}{1+\tau s}$$

$$1 = A(1+\tau s) + Bs \qquad \text{put } s=0$$

$$\boxed{1 = A}$$

$$\text{put } s = -1/\tau$$

$$1 = B\left(-\frac{1}{\tau}\right)$$

$$\boxed{B = -\tau}$$

$$\therefore \boxed{C(s) = \frac{1}{s} - \frac{\tau}{1+\tau s}}$$

Apply inverse L.T on above eqn.

$$C(s) = \frac{1}{s} - \frac{\tau}{1+\tau s}$$

$$C(s) = \frac{1}{s} - \frac{\tau}{\tau(s + 1/\tau)}$$

$$\boxed{c(t) = 1 - e^{-t/\tau}}$$

Steady state error:-

$$e_{ss} = \lim_{s\to 0} s\left[R(s) - C(s)\right]$$

$$= \lim_{s\to 0} s\left[\frac{1}{s} - \frac{1}{s(1+\tau s)}\right]$$

$$= \lim_{s\to 0}\left(1 - \frac{1}{1+\tau s}\right)$$

$$\boxed{e_{ss} = \cancel{\lim}\, 0 //.}$$

Input : unit Ramp signal.

$R(s) = 1/s^2$. $\Rightarrow$ substitute in main eqn.

$$\therefore C(s) = \frac{1}{s^2}\left(\frac{1}{1+\tau s}\right)$$

$$C(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{1+\tau s}$$

$\cancel{1 = AS^2 + AS(\tau s+1) + CS^2}$

$1 = AS(\tau s+1) + B(1+\tau s) + Cs^2$

$\boxed{\text{Put } s=0}$

$\boxed{1 = B}$

$\boxed{\text{Put } s = -1/\tau}$

$1 = C(-1/\tau)^2$

$\boxed{C = \tau^2}$

$\boxed{\text{Put } s=1}$

$1 = A(1+\tau) + B(1+\tau) + C$

$1 = (A+B)(1+\tau) + C$

$1 = (A+1)(1+\tau) + \tau^2$

$\cancel{1} = A\cdot(1+\tau) + \cancel{1} + \tau + \tau^2$

$-\tau(\cancel{1+\tau}) = A(\cancel{1+\tau})$

$\boxed{A = -\tau}$

$\therefore \quad C(s) = -\frac{\tau}{s} + \frac{1}{s^2} + \frac{\tau^2}{1+\tau s}$

$C(s) = -\frac{\tau}{s} + \frac{1}{s^2} + \frac{\tau^2}{\tau(s+1/\tau)}$

Take I.L.T,

$C(t) = -\tau + 1 + \tau e^{-t/\tau}$

$\boxed{C(t) = 1 - \tau(1 - e^{-t/\tau})}$

Steady state error:-

$$e_{ss} = \lim_{s \to 0} s\left[R(s) - C(s)\right]$$

$$= \lim_{s \to 0} s\left[\frac{1}{s^2} - \frac{1}{s^2(\tau s+1)}\right]$$

$$= \lim_{s \to 0} \frac{1}{s}\left(1 - \frac{1}{\tau s+1}\right)$$

$$= \lim_{s \to 0} \frac{1}{\cancel{s}}\left(\frac{\tau \cancel{s}}{1+\tau s}\right)$$

$$\boxed{e_{ss} = \tau}$$ //

Response of first order system

Sno	Signal	input $R(s)$	output $c(t)$	Steady state error 'e_{ss}'
1.	Impulse signal	1	$\frac{1}{\tau} e^{-t/\tau}$ Diff	0
2.	Step signal	$1/s$	$1 - e^{-t/\tau}$ Integ.	0
3.	Ramp signal	$1/s^2$	$t - \tau(1 - e^{-t/\tau})$	τ

$~~1 = AS^2 + AS(\tau S+1) + CS^2~~$

$1 = AS(\tau S+1) + B(1+\tau S) + CS^2$

Put $S=0$

$\boxed{1 = B}$

Put $S = -1/\tau$

$1 = C\left(-1/\tau\right)^2$

$\boxed{C = \tau^2}$

Put $S=1$

$1 = A(1+\tau) + B(1+\tau) + C$

$1 = (A+B)(1+\tau) + C$

$1 = (A+1)(1+\tau) + \tau^2$

$\cancel{1} = A\cdot(1+\tau) + \cancel{1} + \tau + \tau^2$

$-\tau(\cancel{1+\tau}) = A(\cancel{1+\tau})$

$\boxed{A = -\tau}$

$\therefore \quad C(S) = -\frac{\tau}{S} + \frac{1}{S^2} + \frac{\tau^2}{1+\tau S}$

$C(S) = -\frac{\tau}{S} + \frac{1}{S^2} + \frac{\tau^2}{\tau(S+1/\tau)}$

Take I.L.T,

$C(t) = -\tau + 1 + \tau e^{-t/\tau}$

$\boxed{C(t) = 1 - \tau(1 - e^{-t/\tau})}$

Differentiation of step signal ⇒ Impulse s/g.

$$\frac{d}{dt}(1-e^{-t/\tau}) \Rightarrow \frac{1}{\tau}e^{-t/\tau}.$$

Integration of step signal ⇒ Ramp signal.

$$c(t) = \int (1-e^{-t/\tau}) = t + \tau e^{-t/\tau} + C.$$

Assume initial conditions are zero,

$$c(0) = 0 + \tau + C.$$

$$\boxed{C = -\tau}$$

$$\therefore c(t) = \int (1-e^{-t/\tau}) = t + \tau e^{-t/\tau} - \tau \Rightarrow \text{Ramp s/g}.$$

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Measures of performance of Second order system.

The standard form of closed loop transfer function of second order system is given by,

$$T.F = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

where ω_n → Natural undamped freq (rad/sec)

ξ → damping ratio = $\frac{\text{Actual damping}}{\text{critical damping}}$ (no units)

Depend on Damping ratio 'ξ', system can be classified in to four cases:

(i) undamped system : $\xi = 0$

(ii) under damped system : $0 < \xi < 1$

(iii) critically damped system : $\xi = 1$

(iv) over damped system : $\xi > 1$

The characteristic equation of second order system,

$$\boxed{s^2 + 2\xi\omega_n s + \omega_n^2 = 0}$$

It is a Quadratic equation, & root of this eqn,

Quadratic formula $= \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$a = 1, \; b = 2\xi\omega_n, \; c = \omega_n^2$

$\therefore$ roots $s_1, s_2 = \dfrac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2}$

$$\boxed{s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}} \rightarrow \text{roots of second order eqn.}$$

<u>case(i)</u> undamped ($\xi = 0$)

$\boxed{s_1, s_2 = \pm j\omega_n}$ (pure imaginary)

Case (ii) under damped $0 < \xi_\rho < 1$

$$s_1, s_2 = -\xi_\rho \omega_n \pm \omega_n \sqrt{\xi_\rho^2 - 1}$$

$$= -\xi_\rho \omega_n \pm \omega_n \sqrt{(-1)(1-\xi_\rho^2)}$$

$$\boxed{s_1, s_2 = -\xi_\rho \omega_n \pm j\, \omega_n \sqrt{1-\xi_\rho^2}}$$ (complex conjugate).

Case (iii) critically damped $\xi_\rho = 1$

$$\boxed{s_1, s_2 = -\omega_n}$$ (real & equal).

Case iv Over damped $\xi_\rho > 1$

$$\boxed{s_1, s_2 = -\xi_\rho \omega_n \pm \omega_n \sqrt{\xi_\rho^2 - 1}}$$ (real & unequal)

In general, roots of second order s/m $\Big\}$ $\boxed{s_1, s_2 = -\xi_\rho \omega_n \pm \omega_n \sqrt{\xi_\rho^2 - 1}}$

where $\sigma = \xi_\rho \omega_n \Rightarrow$ attenuation &

$\omega_n \sqrt{1-\xi_\rho^2} = \omega_d \Rightarrow$ frequency of damped oscillation (rad/sec).

$\therefore$ roots of Second order System $\Big\}$ $\boxed{s_1, s_2 = -\sigma \pm \omega_d}$ //.

Time Domain response of second order system for unit Impulse signal.

Second order Transfer fn $\boxed{T = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}}$

Input : impulse s/g. $\therefore$ $C(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$

$\hookrightarrow$ Main eqn.

Case (i): undamped s/m ($\xi = 0$) sub in main eqn,

$$C(s) = \frac{\omega_n^2}{s^2 + \omega_n^2} = \omega_n \cdot \frac{\omega_n}{s^2 + \omega_n^2}$$

Taking Inverse L.T,

$$\boxed{c(t) = \omega_n \sin \omega_n t}$$

Case (ii): critically damped s/m ($\xi = 1$) sub in main eqn.

$$C(s) = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s + \omega_n)^2}$$

Taking I.LT

$$\boxed{c(t) = \omega_n^2 \, t \, e^{-\omega_n t}}$$

case(iii): Overdamped system ($\xi_p > 1$) sub in main eqn.

$$C(s) = \frac{\omega_n^2}{s^2 + 2\xi_p \omega_n s + \omega_n^2}$$

Roots $s_1, s_2 = -\xi_p \omega_n \pm \omega_n \sqrt{\xi_p^2 - 1}$.

$$s_1 = -\xi_p \omega_n + \omega_n \sqrt{\xi_p^2 - 1}$$

$$s_2 = -\xi_p \omega_n - \omega_n \sqrt{\xi_p^2 - 1}$$

$$\therefore C(s) = \frac{A}{s - s_1} + \frac{B}{s - s_2} = \frac{\omega_n^2}{s^2 + 2\xi_p \omega_n s + \omega_n^2}.$$

$$\omega_n^2 = A(s - s_2) + B(s - s_1) \qquad \text{sub } \boxed{s = s_1}$$

$$\omega_n^2 = A(s_1 - s_2)$$

$$\boxed{A = \frac{\omega_n^2}{s_1 - s_2}}$$ substitute s_1 & s_2 values.

$$A = \frac{\omega_n^2}{\cancel{-\xi_p \omega_n} + \omega_n \sqrt{\xi_p^2 - 1} \cancel{+ \xi_p \omega_n} + \omega_n \sqrt{\xi_p^2 - 1}}$$

$$= \frac{\omega_n^{\cancel{2}}}{2 \cancel{\omega_n} \sqrt{\xi_p^2 - 1}}$$

$$\boxed{A = \frac{\omega_n}{2\sqrt{\xi_p^2 - 1}}}$$ //.

substitute $s = s_2$,

$$B = \frac{\omega_n^2}{s_2 - s_1}$$

similarly as like A,

$$\boxed{B = \frac{-\omega_n}{2\sqrt{\xi^2-1}}}$$

$$\therefore C(s) = \frac{\omega_n/2\sqrt{\xi^2-1}}{s-s_1} - \frac{\omega_n/2\xi^2-1}{s-s_2}$$

$$C(s) = \frac{\omega_n}{2\sqrt{\xi^2-1}}\left(\frac{1}{s-s_1} - \frac{1}{s-s_2}\right)$$

Take inverse Laplace Transform,

$$\boxed{C(t) = \frac{\omega_n}{2\sqrt{\xi^2-1}}\left(e^{s_1 t} - e^{s_2 t}\right)}$$

Substitute s_1 & s_2,

$$C(t) = \frac{\omega_n}{2\sqrt{\xi^2-1}}\left[e^{(-\xi\omega_n + \omega_n\sqrt{\xi^2-1})t} - e^{(-\xi\omega_n + \omega_n\sqrt{\xi^2-1})t}\right]$$

$$C(t) = \frac{\omega_n}{2\sqrt{\xi^2-1}}\left[e^{-\xi\omega_n t}\left(e^{(\omega_n\sqrt{\xi^2-1})t} - e^{(-\omega_n\sqrt{\xi^2-1})t}\right]$$

$$\boxed{C(t) = \frac{\omega_n}{\sqrt{\xi^2-1}} e^{-\xi\omega_n t}\left[\sinh \omega_n\sqrt{\xi^2-1}\right]t}$$

$$\left(\because \frac{e^{\theta}-e^{-\theta}}{2} = \sinh\theta\right)$$

case(iv): underdamped system ($0 < \xi_p < 1$)

Roots: $S_1, S_2 \Rightarrow -\xi_p \omega_n \pm j\omega_n \sqrt{1-\xi_p^2}$.

$$S_1 = -\xi_p \omega_n + j\omega_n\sqrt{1-\xi_p^2}$$

$$S_2 = -\xi_p \omega_n - j\omega_n\sqrt{1-\xi_p^2}.$$

$$C(S) = \frac{\omega_n^2}{S^2+2\xi_p\omega_n S+\omega_n^2} = \frac{\omega_n^2}{(S-S_1)(S-S_2)}$$

$$\frac{\omega_n^2}{(S-S_1)(S-S_2)} = \frac{A}{S-S_1} + \frac{B}{S-S_2}.$$

$$\omega_n^2 = A(S-S_2) + B(S-S_1) \qquad \text{if } (S=S_1)$$

$$\omega_n^2 = A(S_1-S_2)$$

$$A = \frac{\omega_n^2}{S_1-S_2} = \frac{\omega_n^{\cancel{2}}}{2j\cancel{\omega_n}\sqrt{1-\xi_p^2}} = \boxed{\frac{\omega_n}{2j\sqrt{1-\xi_p^2}} = A}$$

|||ly $\boxed{B = \frac{-\omega_n}{2j\sqrt{1-\xi_p^2}}}$.

$$\therefore C(S) = \frac{\omega_n/2j\sqrt{1-\xi_p^2}}{S-S_1} - \frac{\omega_n/2j\sqrt{1-\xi_p^2}}{S-S_2}.$$

$$C(s) = \frac{\omega_n}{2j\sqrt{1-\xi^2}}\left(\frac{1}{s-s_1} - \frac{1}{s-s_2}\right)$$

Take I.L.T

$$c(t) = \frac{\omega_n}{2j\sqrt{1-\xi^2}}\left(e^{s_1 t} - e^{+s_2 t}\right)$$

$$c(t) = \frac{\omega_n}{2j\sqrt{1-\xi^2}}\left(e^{(-\xi\omega_n + j\omega_n\sqrt{1-\xi^2})t} - e^{(-\xi\omega_n - j\omega_n\sqrt{1-\xi^2})t}\right)$$

$$c(t) = \frac{\omega_n}{2j\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \cdot \left[e^{(j\omega_n\sqrt{1-\xi^2})t} - e^{(-j\omega_n\sqrt{1-\xi^2})t}\right]$$

$$c(t) = \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin\left[\omega_n\sqrt{1-\xi^2}\right]t \qquad \left(\because \frac{e^{j\theta} - e^{-j\theta}}{2j} = \sin\theta\right.$$

$$\boxed{c(t) = \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin\left(\omega_n\sqrt{1-\xi^2}\right)t}$$

Steady state error:- $e_{ss} = \lim_{s\to 0} s\left[R(s) - C(s)\right]$

$$e_{ss} = \lim_{s\to 0} s\left[1 - \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}\right]$$

$$e_{ss} = \lim_{s\to 0} s\left[1 - \frac{\omega_n^2}{\omega_n^2}\right] = 0$$

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Time Response of second order system for unit step signal.

T.F of a second order system } $\frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2+2\xi_p\omega_n S+\omega_n^2}$

unit step signal $\boxed{R(S) = 1/S}$

$$\therefore \ C(S) = \frac{1}{S}\left(\frac{\omega_n^2}{S^2+2\xi_p\omega_n S+\omega_n^2}\right)$$

case (i) undamped, $\xi_p = 0$.

$$C(S) = \frac{1}{S}\left(\frac{\omega_n^2}{S^2+\omega_n^2}\right) = \frac{A}{S} + \frac{BS+C}{S^2+\omega_n^2}$$

$$\omega_n^2 = A(S^2+\omega_n^2) + (BS+C)(S)$$

$S=0$

$$\omega_n^2 = A\omega_n^2 \Rightarrow \boxed{A=1}$$

Expand the eqn,

$$\omega_n^2 = AS^2 + A\omega_n^2 + BS^2 + CS$$

Equating S^2 terms $\Rightarrow$ $0 = A+B$ $\therefore \boxed{B=-1}$

Equating S term $\Rightarrow \boxed{0 = C}$

$$\therefore \ C(S) = \frac{1}{S} + \left(\frac{-S}{S^2+\omega_n^2}\right) = \frac{1}{S} - \frac{S}{S^2+\omega_n^2}$$

Taking Inverse L.T,

$$\boxed{C(t) = 1 - \cos\omega_n t}$$

case ii critically damped system ($\xi = 1$)

$$C(S) = R(S)\left(\frac{\omega_n^2}{S^2 + 2\xi\omega_n S + \omega_n^2}\right)$$

$$C(S) = \left(\frac{1}{S}\right)\left(\frac{\omega_n^2}{S^2 + 2\omega_n S + \omega_n^2}\right) = \left(\frac{1}{S}\right)\left(\frac{\omega_n^2}{(S+\omega_n)^2}\right)$$

Apply partial fraction.

$$C(S) = \frac{1}{S}\left(\frac{\omega_n^2}{(S+\omega_n)^2}\right) = \frac{A}{S} + \frac{B}{S+\omega_n} + \frac{C}{(S+\omega_n)^2}$$

$$\omega_n^2 = A(S+\omega_n)^2 + BS(S+\omega_n) + CS.$$

$$\omega_n^2 = A\omega_n^2 \quad \therefore \boxed{A = 1} \qquad \text{put } S=0$$

Expand the equation.

$$\omega_n^2 = AS^2 + A\omega_n^2 + 2AS\omega_n + BS^2 + BS\omega_n + CS.$$

Equating s^2 term

$$0 = A + B \quad \therefore \boxed{B = -1}$$

Equating s term.

$$0 = 2A\omega_n + B\omega_n + C.$$

$$0 = 2\omega_n - \omega_n + C$$

$$\boxed{C = -\omega_n}$$

$$\boxed{A = 1;\ B = -1;\ C = -\omega_n}$$

$$C(s) = \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}$$

Take inverse Laplace transform.

$$\boxed{c(t) = 1 - e^{-\omega_n t} - \omega_n t\, e^{-\omega_n t}}$$

Case (iii) Overdamped system ($\xi > 1$)

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)}$$

roots $s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$.

$$\therefore C(s) = \frac{\omega_n^2}{s(s-s_1)(s-s_2)}$$

Taking partial fraction.

$$C(s) = \frac{\omega_n^2}{s(s-s_1)(s-s_2)} = \frac{A}{s} + \frac{B}{s-s_1} + \frac{C}{s-s_2} \rightarrow (1)$$

$$\omega_n^2 = A(s-s_1)(s-s_2) + Bs(s-s_2) + Cs(s-s_1).$$

$$\omega_n^2 = A(-s_1)(-s_2) \cdot \quad \therefore \boxed{A = \frac{\omega_n^2}{s_1 s_2}} \qquad \text{put } s=0$$

Substitute s_1, s_2 values in A eqn

$$A = \frac{\omega_n^2}{(-\xi\omega_n + \omega_n\sqrt{\xi^2-1})(-\xi\omega_n - \omega_n\sqrt{\xi^2-1})}$$

$$A = \frac{\omega_n^2}{\xi^2\omega_n^2 - \omega_n^2(\xi^2-1)} \qquad (\because (a+b)(a-b) = a^2-b^2)$$

$$A = \frac{\cancel{\omega_n^2}}{\cancel{\xi^2\omega_n^2} - \cancel{\xi^2\omega_n^2} + \cancel{\omega_n^2}} = 1. \quad \therefore \boxed{A=1}$$

Put $s = s_1$, in partial fraction equation.

$$\omega_n^2 = B\, s_1(s_1 - s_2)$$

$$\boxed{B = \frac{\omega_n^2}{s_1(s_1-s_2)}}$$

substitute s_1, s_2 values in B eqn.

$$B = \frac{\omega_n^2}{(-\xi\omega_n + \omega_n\sqrt{\xi^2-1})(\cancel{-\xi\omega_n} + \omega_n\sqrt{\xi^2-1} + \cancel{\xi\omega_n} + \omega_n\sqrt{\xi^2-1})}$$

$$B = \frac{\omega_n^2}{(-\xi\omega_n + \omega_n\sqrt{\xi^2-1})(2\omega_n\sqrt{\xi^2-1})}$$

$$B = \frac{1}{(-\xi + \sqrt{\xi^2-1})(2\sqrt{\xi^2-1})} \qquad \therefore \boxed{B = \frac{1}{2\sqrt{\xi^2-1}\,(-\xi + \sqrt{\xi^2-1})}}$$

substitute $s = s_2$ in partial fraction eqn.

$$\omega_n^2 = C\, s_2(s_2 - s_1)$$

$$\boxed{C = \frac{\omega_n^2}{s_2(s_2-s_1)}}$$

substitute s_1, s_2 values in C eqn.