Polar plot

Polar plot of a sinusoidal T. F $G(j\omega)$ is a plot of the $|G(j\omega)|$ Vs $\angle G(j\omega)$ as ω is varied from 0 to ∞

Thus the polar plot is the locus of vertex $|G(j\omega)| \angle G(j\omega)$ as ω is varied from 0 to ∞

It is also called Nyquist plot.

Polar plot → polar Sheet

↳ concentric circles → magnitude

↳ Radial lines → phase angle.

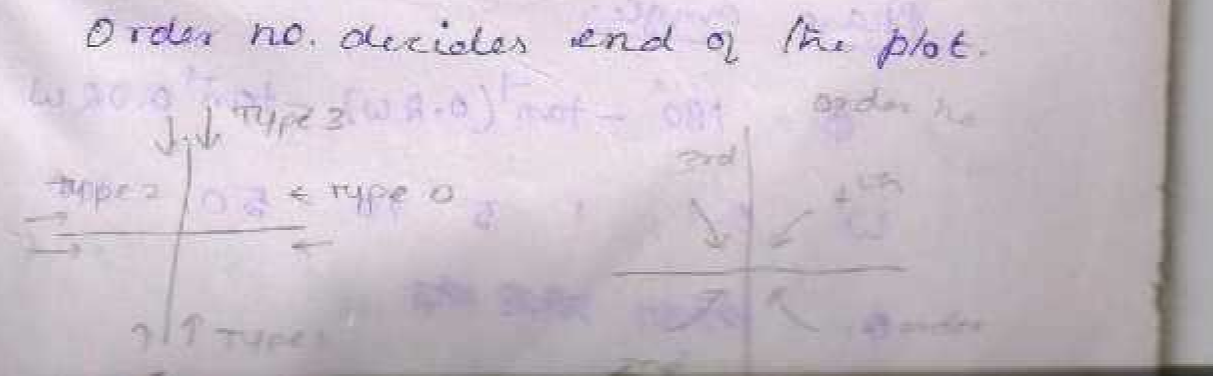
Or otherwise, polar plot in graph sheet.

The ω value can be chosen such that corner frequency or around corner frequency.

For minimum transfer function (only with poles) we can draw the polar plot easily from type number and order number

Type number → Start of the plot.

Order no. decides end of the plot.



The closed loop Transfer function of the system is given by,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

The sinusoidal T.F is obtained by replacing by $j\omega$

$$\frac{C(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1+G(j\omega)H(j\omega)}$$

$$\frac{C(j\omega)}{R(j\omega)} = M(j\omega) = M\angle\alpha$$

where, $M \rightarrow$ Magnitude of CL T.F

$\alpha \rightarrow$ phase angle of CL T.F

* The M & α of CL system are function of frequency (ω).

* The sketch of M & α of CL system with respect to ω is closed loop frequency response plot.

* The M & α of CL system for various value of ω can be evaluated analytically or graphically.

* Two Graphical methods are available to determine the closed loop frequency from OL

⇒ Nichols chart

M & N circles:

The magnitude of CLTF with unity feedba[ck] can be shown to be in the form of circles for every values of 'M'. These circles are called M-circles

If the α is phase angle of CLTF with unity feedback then it can be shown that $\tan\alpha$ will be in the form of circles for every values of α. These circles are called N-circles.

M-circles proof:

Consider the CLTF of unity feedback syst[em]

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

put $s = j\omega$

$$\frac{C(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1+G(j\omega)}$$

Let $G(j\omega) = x + jy$

where $x \to$ Real part of $G(j\omega)$

$y \to$ Imaginary part of $G(j\omega)$

Let M = Magnitude of $\frac{C(j\omega)}{R(j\omega)}$

$$M = \frac{\sqrt{x^2+y^2}}{\sqrt{(1+x)^2+y^2}}$$

On squaring,

$$M^2 = \frac{x^2+y^2}{(1+x)^2+y^2}$$

$$M^2(1+x)^2 + y^2 = x^2+y^2$$

$$M^2(1+x^2+2x) + y^2 = x^2+y^2$$

$$M^2 + M^2x^2 + 2M^2x + M^2y^2 - x^2 - y^2 = 0$$

$$x^2(M^2-1) + M^2 2x + M^2 + y^2(M^2-1) = 0 \quad \text{--- (1)}$$

When M = 1 equ (1) represent a straight line.

$$2x + 1 = 0.$$

$$\boxed{x = -1/2}$$

Hence, when M = 1, equ (1) ~~po~~ represent a straight line passing through $x = -1/2$, $y = 0$

When $M \neq 1$, The equ (2) represent a family of circles.

Consider equ (1).

$$x^2 + \frac{M^2}{M^2-1} 2x + \frac{M^2}{M^2-1} + y^2 = 0.$$

Add $\frac{M^2}{(M^2-1)^2}$ on both sides.

$$x^2 + \frac{M^2}{(M^2-1)} 2x + \frac{M^2}{(M^2-1)} + y^2 + \frac{M^2}{(M^2-1)^2} = \frac{M^2}{(M^2-1)^2}$$

$$x^2 + \frac{M^2}{M^2-1} 2x + \frac{M^2(M^2-1)+M^2}{(M^2-1)^2} + y^2 = \frac{M^2}{(M^2-1)^2}$$

$$\left(x^2 + \frac{M^2}{M^2-1} 2x + \frac{M^4}{(M^2-1)^2}\right) + y^2 = \frac{M^2}{(M^2-1)^2}$$

$$\left[x + \frac{M^2}{M^2-1}\right]^2 + y^2 = \frac{M^2}{(M^2-1)^2} \longrightarrow (2)$$

The equation of circle with centre at (x_1, y_1) & radius 'r' is given by.

$$(x-x_1)^2 + (y-y_1)^2 = r^2 \text{ --- } (3)$$

Compare equ (2) & (3) it can be conclude that equ (2) represent a family of circles with centre at $\left(\frac{-M^2}{M^2-1}, 0\right)$ and with radius $r = \frac{M}{M^2-1}$

The circles given by equ (2) are called M-circles.

magnitude circle becomes a point at (0, 0).

$y_1 = 0$

$r = \frac{M}{M^2-1} = 0.$

When $M = \infty$

$x_1 = \frac{-M^2}{M^2-1} \approx \frac{-M^2}{M^2} = -1$

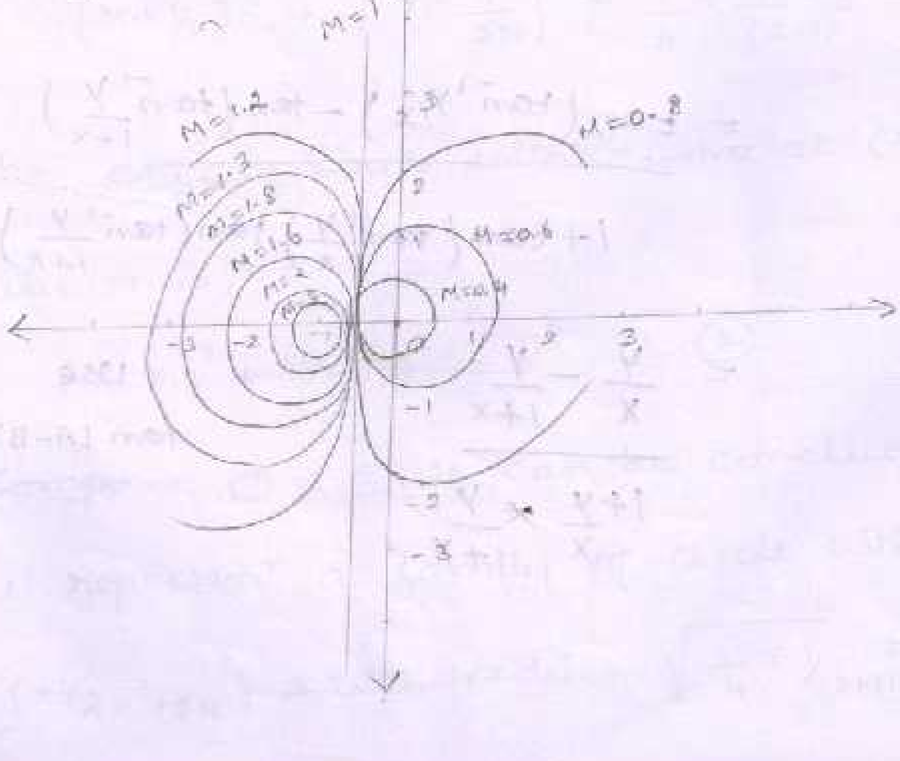
$y_1 = 0$

$r = \frac{M}{M^2-1} \approx \frac{M}{M^2} = \frac{1}{M} = 0.$

Hence when $M = \infty$, the magnitude circle becomes a point at $(-1, 0)$

From the above analysis it is clear that the magnitude of CLTF in the form of circles.

When $M \neq 1$ & when $M = 1$, the magnitude is a straight line passing through $(-1/2, 0)$



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

Put $s = j\omega$

$$\frac{C(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1+G(j\omega)}$$

Let $G(j\omega) = x + jy$ where $x \rightarrow$ Real part of G(

$y \rightarrow$ Imaginary part of G(

$$\frac{C(j\omega)}{R(j\omega)} = \frac{x+jy}{1+x+jy} = \frac{\sqrt{x^2+y^2}\ \tan^{-1} y/x}{\sqrt{(1+x)^2+y^2}\ \tan^{-1} y/_{1+x}}$$

Let $\alpha = Arg\left[\frac{C(j\omega)}{R(j\omega)}\right]$

$$\alpha = \tan^{-1} y/x - \tan^{-1}\frac{y}{1+x}$$

Let $N = \tan\alpha$

$$\therefore N = \tan\left[\tan^{-1} y/x - \tan^{-1}\frac{y}{1+x}\right]$$

$$= \frac{\tan(\tan^{-1} y/x) - \tan(\tan^{-1}\frac{y}{1+x})}{1+\tan(\tan^{-1}\frac{y}{x})\tan(\tan^{-1}\frac{y}{1+x})}$$

$$= \frac{\frac{y}{x} - \frac{y}{1+x}}{1+\frac{y}{x}\times\frac{y}{1+x}}$$

Use $\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B}$

$$\frac{x(1+x)+y^2}{x(1+x)}$$

$$= \frac{y}{y^2+x(1+x)}$$

$$\therefore N = \frac{y}{x^2+y^2+x}$$

On rearranging we get,

$$x+x^2+y^2 = \frac{y}{N}$$

$$x^2+y^2+x-\frac{y}{N} = 0$$

Add the term $\frac{1}{4}+\left(\frac{1}{2N}\right)^2$ on both sides.

$$x+x^2+y^2-\frac{y}{N}+\frac{1}{4}+\left(\frac{1}{2N}\right)^2 = \frac{1}{4}+\left(\frac{1}{2N}\right)^2$$

$$\left(x^2+\frac{1}{4}+x\right)+\left(y^2+\frac{1}{(2N)^2}-\frac{y}{N}\right) = 1/4+\frac{1}{(2N)^2}$$

$$(x+1/2)^2+\left(y-\frac{1}{2N}\right)^2 = \frac{1}{4}+\frac{1}{(2N)^2} \quad \text{—} (1)$$

The equ. of circle with centre at (x_1, y_1) & radius r is,

$$(x-x_1)^2+(y-y_1)^2 = r^2 \quad \text{—} (2)$$

Compare (1) & (2) it can be conclude that the equ(1) represent a family of circle with centre at $(-1/2, 1/2N)$ & with radius $\sqrt{\frac{1}{4}+1/(2N)^2}$

The equ of N-circles is satisfied at 2 points (0,0) & (-1,0). Hence the N. circle passes through these 2 points for all value of α

Consider the equn of 'N' circle when $x=0, y=0$.

$$\left(x+\tfrac{1}{2}\right)^2+\left(y-\tfrac{1}{2N}\right)^2=\frac{1}{4}+\frac{1}{(2N)^2}$$

$$\left(\frac{1}{2}\right)^2+\left(-\tfrac{1}{2N}\right)^2=\frac{1}{4}+\frac{1}{(2N)^2}$$

$$\frac{1}{4}+\frac{1}{4N^2}=\frac{1}{4}+\frac{1}{(2N)^2}$$

Consider the equ of 'N' circles when $x=-1, y=0$.

$$\left(x+\tfrac{1}{2}\right)^2+\left(y-\tfrac{1}{2N}\right)^2=\frac{1}{4}+\frac{1}{(2N)^2}$$

$$\left(-1+\tfrac{1}{2}\right)^2+\left(\frac{-1}{2N}\right)^2=\frac{1}{4}+\frac{1}{4N^2}$$

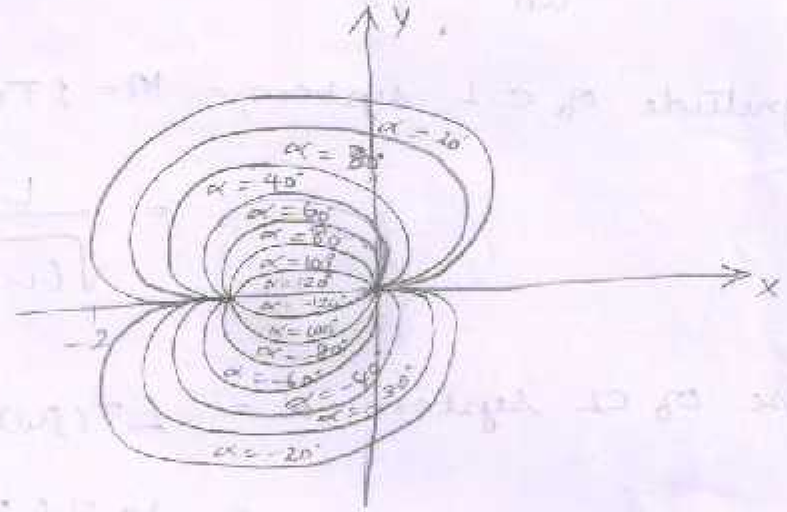
$$\frac{1}{4}+\frac{1}{4N^2}=\frac{1}{4}+\frac{1}{4N^2}$$

$\Rightarrow$ The above analysis show that the equ. of 'N' circles is satisfied at point (0,0) & (-1,0)

When $\alpha=180°$, the circle becomes a straight line passing through real axis.

It is also observed that the circle for $\alpha=0°$ above the real axis will be a part of circle for $\alpha=0°$ below the real axis is shown below

The family of N-circles as shown in fig.



Correlation between Time & Frequency response:

Consider the standard form of 2^{nd} order syst.

$$\frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2 + 2\xi\omega_n S + \omega_n^2}$$

where $\xi \rightarrow$ Damping factor

$\omega_n \rightarrow$ undamped natural frequency.

The sinusoidal T.F of the system is obtained by let $S = j\omega$

$$\frac{C(j\omega)}{R(j\omega)} = T(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\xi\omega_n(j\omega) + \omega_n^2}$$

$$= \frac{\omega_n^2}{-\omega^2 + j2\xi\omega_n\omega + \omega_n^2}$$

$$= \frac{\omega_n^2}{\omega_n^2\left(\frac{-\omega^2}{\omega_n^2} + j2\xi\frac{\omega}{\omega_n} + 1\right)}$$

$$= \frac{1}{(1-u^2)+j2\xi u}$$

where $u = \frac{\omega}{\omega_n}$ is the normalized frequency.

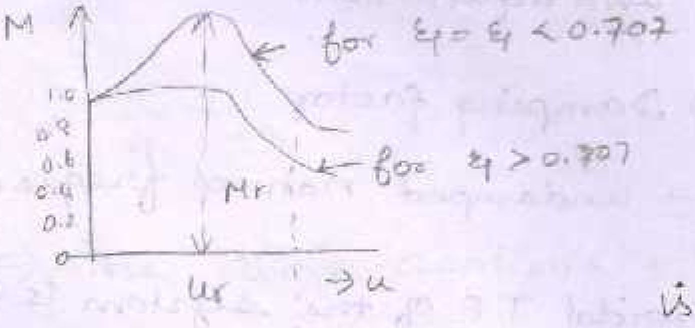
Magnitude of C.L system, $= M = |T(j\omega)|$

$$= \frac{1}{\sqrt{(1-u^2)^2+(2\xi u)^2}}$$

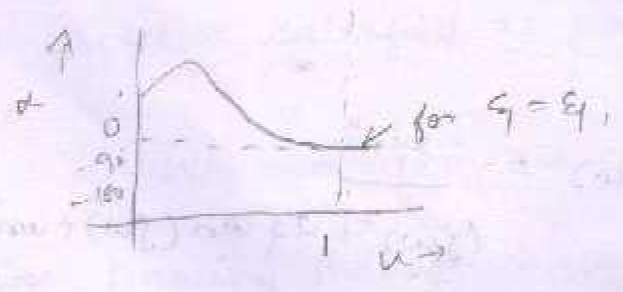
Phase of CL system $\alpha = \angle T(j\omega)$

$$= -\tan^{-1}\left(\frac{2\xi u}{1-u^2}\right)$$

The magnitude & phase angle characteristics for normalised frequency u, for certain values of ξ are shown in fig.



The freq. at which M has a peak value is known as resonance frequency.



The peak value of the mag. is the resonant peak Mr.

$\Rightarrow$ Let ω_r be the resonant frequency & $u_r = \frac{\omega_r}{\omega_n}$ be the normalized resonant frequency.

$$M_r = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

$$\omega_r = \omega_n\sqrt{1-2\xi^2}$$

output of the system is bounded (finite) for the bounded input thus the system is stable system.

If the output is unbounded for the given unbounded input thus the system is unstable system.

If the output has constant amplitude, oscillation, the system may be stable or unstable system. Then the system is known as limitedly ~~search~~ stable system.

Routh - Herwitz criterion :

- → Hurwitz Criterion → determinant form.
- → ~~Row~~ Routh - ~~Array~~ formulation.

2)

& equating $\frac{dM}{du}$ to Zero.

Cont - I

Completed

29/11/12

stability of the system representing the Characteristic equation.

$s^4+8s^3+18s^2+16s+5=0$ comment the locat[ion] of root characteristics equation

Sol

Routh array.

s^4	1	18	5
s^3	8	16	0
s^2	16	5	0
s^1	13.5	0	0
s^0	5	0	0

$\frac{8\times18-1\times16}{8}=16$

$\frac{8\times5-1\times0}{8}=5$

$\frac{16\times16-8\times5}{16}=13$[.5]

$\frac{13.5\times5-0}{13.5}=5$

There is no sign changes in the 1st c[olumn] of routh array.

$\therefore$ The system is stable.

comment: The every values can be locat[ed] in the ~~locatin~~ left ~~half~~ ~~plane~~ side of s plan[e]

2) Construct routh array & determine the stabil[ity] of the sys whose chas. equ is $s^6+2s^5+8s^4+12s^3$ $+20s^2+16s$ [...]

also determine the no. of roots lies in left & half

s^6	1	8	20	16
s^5	2	12	16	
s^4	2	12	16	
s^3	0	0	0 → special case	
s^3	8	24		
s^2	6	16		
s^1	2.67	0		
s^0	16	0		

$\frac{2\times 8 - 1\times 12}{2} = 2$

$\frac{2\times 20 - 16}{2} = 12$

$\frac{2\times 16 - 0}{2} = 16$

$\boxed{A = 2s^4 + 12s^2 + 16}$

$\frac{dA}{ds} = 8s^3 + 24s$

$2s^4 + 12s^2 + 16$

$s^2 = x$

$2x^2 + 12x + 16 = 0$

$x^2 + 6x + 8 = 0$

$(x+4)(x+2) = 0$

$x = -4, -2$

$s = \pm\sqrt{x}$

$= \pm\sqrt{-2}, \pm\sqrt{-4}$

$= \pm j\sqrt{2}, \pm j2$

The roots are

$j\sqrt{2}, -j\sqrt{2}, j2, -j2$

Remaining roots are in <u>stable</u>.

∴ the system is <u>limitedely stable</u>.

The roots lying on the imaginary axis

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3. Construct Routh array & determine the stability of the system represented by the char. equn. $s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5 = 0$. Comment on the location of the roots of characteristics equn.

$s^3 \quad 0 \quad -2$

$s^2 \quad \frac{2\epsilon_1+2}{\epsilon_1} \quad 5$

$s^1 \quad \frac{\frac{(2\epsilon_1+2)(-2)-5\epsilon_1}{\epsilon_1}}{\frac{2\epsilon_1+2}{\epsilon_1}}$

$$\frac{\frac{-4\epsilon_1-4-5\epsilon_1}{\epsilon_1}}{\frac{2\epsilon_1+2}{\epsilon_1}}$$

$$\frac{\frac{-4\epsilon_1-4-5\epsilon_1^2}{\epsilon_1}}{\frac{2\epsilon_1+2}{\epsilon_1}}$$

$s^1 \quad -\left[\frac{5\epsilon_1^2+4\epsilon_1+4}{2\epsilon_1+2}\right]$

$s^0 \quad 5$

Let $\epsilon_1 = 0$,

s^5	1	2	3
s^4	1	2	5
s^3	0	-2	
s^2	∞	5	
s^1	-2		
s^0	5		

$\therefore$ By examine the 1^{st} column of routh array there is two sign changes.

The system lies in right half of s plane.

the system is unstable.

the remaining roots are in stable state.

$9s^5 - 20s^4 + 10s^3 - s^2 - 9s - 10 = 0$. Comment on the location of roots of characteristic equn.

Sol

s^5	9	10	-9
s^4	-20	-1	-10
s^3	9.55	-13.5	
s^2	-29.3	-10	
s^1	-16.8		
s^0	-10		

In the first column of the routh array, there is three sign changes.

∴ The roots are lying on the right side of splane.

The remaining roots are lying in the left side of s plane.

∴ The system is unstable.

Q The char. polynomial of the system is $s^7 + 9s^6 + 24s^5 + 24s^4 + 24s^3 + 24s^2 + 23s + 15 = 0$. Det the location of roots of on splane and find stability of system

s^5	21.3	21.3	21.3
s^4	15	15	15
s^3	0	0	0
s^3	4	2	
s^2	7.5	15	
s^1	-6		
s^0	15		

$A = 15s^4 + 15s^2 + 15$

$s^2 = x \quad s^4 + s^2 + 1$

$= 15x^2 + 15x + 15$

$s = \sqrt{x} \qquad x^2 + x + 1 = 0$

$\frac{dA}{ds} = 4s^3 + 2s$

$(ax^2+bx+c) \quad a = 1 \quad b = 1 \quad c = 1$

$x = \frac{-1 \pm \sqrt{1-4}}{2}$

$= \frac{-1 \pm \sqrt{-3}}{2}$

$s = \frac{-1 \pm j\sqrt{3}}{2}$

$x = \frac{-1 \pm j\sqrt{3}}{2}$

There are 2 sign changes.

2 roots lying on the Right side of S-plane.

The system is unstable.

$S = \pm\sqrt{x}$

$S = \pm\sqrt{1\angle 120^\circ}$ & $S = \pm\sqrt{1\angle -120^\circ}$

$= \pm 1 \angle 120^\circ/2$ & $S = \pm\sqrt{1} \angle -120^\circ/2$

$S = \pm 1 \angle 60^\circ \qquad S = \pm 1 \angle -60^\circ$

$S = \pm(0.5 + j0.866) \qquad S = \pm(0.5 - j0.866)$

$S = 0.5 + j0.866 \qquad S = 0.5 - j0.866$

$S = -0.5 - j0.866 \qquad S = -0.5 + j0.866$

Remaining 5 roots are placed on the L.H.S.

→ The ~~stable~~ system is unstable.

Sol

Unity feedback.

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$= \frac{G(s)}{1+G(s)} = \frac{\frac{K}{s(s+1)(s+2)}}{1+\frac{K}{s(s+1)(s+2)}}$$

$$= \frac{\frac{K}{s(s+1)(s+2)}}{\frac{s(s+1)(s+2)+K}{s(s+1)(s+2)}}$$

$$\frac{C(s)}{R(s)} = \frac{K}{K+s(s+1)(s+2)} = \frac{K}{K+s(s^2+s+2s+2}$$

$$= \frac{K}{K+s^3+s^2+2s^2+2s}$$

$$\frac{C(s)}{R(s)} = \frac{K}{K+s^3+3s^2+2s}$$

∴ The char. equ is

$$s^3+3s^2+2s+K=0$$

s^3	1	2
s^2	3	K
s^1	$\frac{6-K}{3}$	
s^0	K	

$6 - K > 0$

$6 - 0 > K$

$6 > K$

$\therefore$ Range of K.

$$\boxed{0 < K < 6}$$

Location of Roots on the S-plane for stability:

The CL TF $M(S) = \dfrac{b_0 S^m + b_1 S^{m-1} + b_2 S^{m-2} + \dots + b_{m-1} S + b_m}{a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_{n-1} S + a_n}$

$$= \frac{(S+z_1)\,(S+z_2) + \dots + (S+z_n)}{(S+P_1)\,(S+P_2) \dots (S+P_n)}$$

The roots of numerator polynomial $z_1, z_2 \dots z_m$ are Zeros.

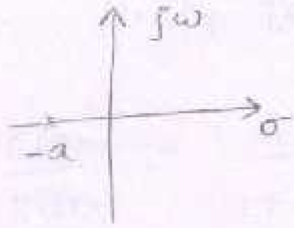
The roots of denominator polynomial $P_1, P_2 \dots P_m$ are poles.

$\Rightarrow$ Denominator polynomial is the characteristic equation & so the poles are the roots of char. eqn.

By partial fraction expansion,

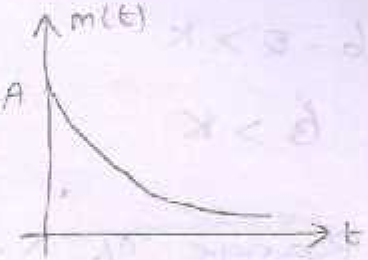
$$M(S) = \frac{A_1}{S+P_1} + \frac{A_2}{S+P_2} + \dots + \frac{A_n}{S+P_n}$$

1) $M(s) = \dfrac{A}{s+a}$



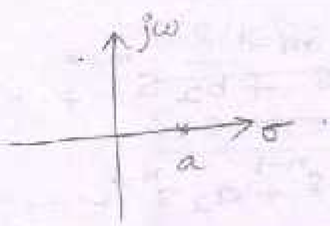
Root on −ve real axis

$m(t) = L^{-1}\left(A/_{s+a}\right) =$



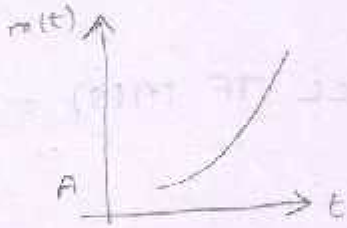
Response is exponentially decaying.

2) $M(s) = \dfrac{A}{s-a}$



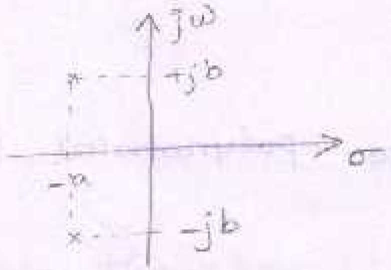
Root on +ve real axis

$m(t) = L^{-1}\left(\dfrac{A}{s-a}\right) = Ae^{at}$



Response is exponentially ↑ing.

3) $M(s) = \dfrac{A}{s+a+jb} + \dfrac{A}{s+a-jb}$



Complex conjugate on left half of s-plane

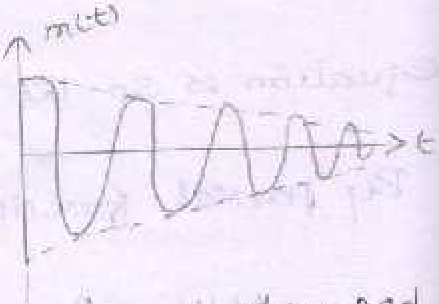
$M(t) = L^{-1}\left[\dfrac{A}{s+a+jb} + \dfrac{A}{s+a-jb}\right]$

$= Ae^{-(a+jb)t} + Ae^{-(a-jb)t}$

$= Ae^{-at}\left[e^{-jbt} + e^{jbt}\right]$

$= 2Ae^{-at}\cos bt$

$= 2Ae^{-at}\sin(bt+\phi)$



Response is damped sinusoidal.