Components of a DSP System.

I/p Analog s/l → ADC —(I/p Digital s/l)→ Digital System —(o/p digital s/m)→ DAC → o/p Analog s/l.

The Processing of s/l are basically spectrum analysis to determine the various frequency components of a signal and filtering the s/l to extract the reqd. freq. component of the s/l.

The digital system can be a specially designed programmable hardware for DSP or an algorithm/Software running on a general Purpose digital system like personal Computer.

Advantages:-

- ✓ Digital hardware are compact, reliable, less expensive and Programmable.
- ✓ System can be easily upgraded/modified.

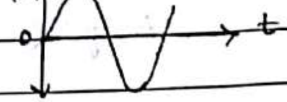
Signal:-

Any Physical Phenomenon that Conveys or carries some information.

Signal
- Analog
  - Continuous time
- Discrete
  - Discrete time

Analog s/e:-

When a s/e is defined continuously for any value of an independent variable it is called an analog (or) continuous s/e.



When the dependent variable of an analog s/e is time, it is called a continuous time s/e and it is denoted as "$x(t)$".

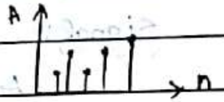
Digital s/e:-

When a s/e is defined for discrete intervals of an independent variable, it is called a discrete signal.

When the dependent variable of a discrete s/e is time, it is called discrete time s/e and it is denoted by "$x(n)$".

System:-

Any process that exhibits cause and effect relation. (or) It is an interconnection of components. It is a physical device that performs an operation on an I/p s/e and produces another s/e as output.

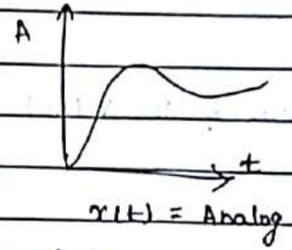


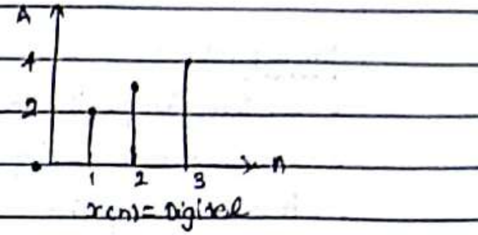
Discrete time system:-

A system which can process a discrete time s/e is called a discrete time system, and so the I/p & o/p s/e of a discrete time system are discrete time signals.

Signal:-

May be any physical quantity such as temperature, pressure with respect to independent variable such as time, space.





Processing:-

Any operation performed on the s/e to extract the information (or) to modify the information according to the requirements. Eg:- Filtering.

Amplification:-

Increasing the s/e strength.

Attenuation:-

Decreasing (or) suppressing the strength of s/e.

Sampling:-

Analog to Digital Conversion. ($f_s \geq 2f_m$)

Elements of DSP:-

Analog I/p → LPF → Sampling & Hold circuit → ADC → DSP → DAC → Reconstruction filter → Analog o/p

- LPF: To limit max. I/p Freq
- Sampling & Hold circuit: Provides constant analog I/p Vge.
- ADC: Conversion

Need for DSP:-

- ✓ Ease of storage & transportation
- ✓ flexibility
- ✓ Repeatability
- ✓ less noise
- ✓ Greater Accuracy
- ✓ Applicable to VLF (ECG, EMG, EEG)

Drawbacks:-

- ✓ System Complexity Increases (becoz of ADC & DAC)
- ✓ Sampling rate is high.
- ✓ Power Consumption

Discrete time signals:-

Generation of Discrete Time Signals:-

✓ Generate a set of numbers and arrange them as a sequence.

Eg:- $x(n) = 2n \; ; \; 0 \le n \le N$

✓ Evaluation of a numerical recursion relation will generate a discrete signal.

Eg:- $x(n) = 0.2\, x(n-1)$ with initial condition $x(0) = 1$,

gives the sequence, $x(n) = 0.2^n \; ; \; 0 \le n < \infty$

when $n = 0$; $x(0) = 1$ (Initial condition) $= 0.2^0$

when $n = 1$; $x(1) = 0.2\, x(1-1) = 0.2\, x(0) = 0.2 = 0.2^1$

$\vdots$

$\therefore x(n) = 0.2^n \; ; \; 0 \le n < \infty$

✓ Uniformly sampling a continuous time signal and using the amplitudes of the samples to form a sequence

let, $x(t)$ = continuous time s/e

Now, Discrete s/e, $x(nT) = x(t)\big|_{t=nT} \; ; \; -\infty < n < \infty$

$T \rightarrow$ Sampling Interval.

Representation of Discrete time Signals:-

- ✓ Functional Representation
- ✓ Graphical Representation
- ✓ Tabular Representation
- ✓ Sequence Representation

Functional Representation:-

The signal is represented as a mathematical equation.

$$x(n) = \begin{cases} -0.5 \; ; & n = -2 \\ 1.0 \; ; & n = -1 \\ -1.0 \; ; & n = 0 \\ 1.0 \; ; & n = 1 \\ -1.0 \; ; & n = 2 \\ 2.0 \; ; & n = 3 \\ 0 \; ; & \text{other } n. \end{cases}$$

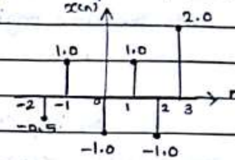


Fig. 1

Graphical Representation:-

The s/e is represented in a 2-D plane. The independent variable is represented in the horizontal axis and the value of the s/e is represented in the vertical axis as shown in fig 1.

Tabular Representation:-

Two rows of a table are used to represent a discrete time s/e. In the first row, the independent variable "n" is tabulated and in the second row the value of the s/e for each value of "n".

n	......	-2	-1	0	1	2	3	...
x(n)	......	-0.5	1.0	-1.0	1.0	-1.0	2.0	...

Sequence Representation:-

The discrete time s/e is represented as a 1-D array.

Example:-

An infinite duration discrete time s/e is represented with the time origin, n=0, indicated by the symbol ↑

$$x(n) = \{\ldots\ldots -0.5, 1.0, \underset{\uparrow}{-1.0}, 1.0, -1.0, 2.0 \ldots\}$$

An infinite duration discrete time signal that satisfies the condition $x(n) = 0$ for $n < 0$ is represented as,

$$x(n) = \{\underset{\uparrow}{-1.0}, 1.0, -1.0, 2.0, \ldots\} \text{ or } x(n) = \{-1.0, 1.0, -1.0, 2.0 \ldots\}$$

A finite duration discrete time s/e with the time origin, n=0, indicated by the symbol ↑ is represented as

$$x(n) = \{-0.5, 1.0, \underset{\uparrow}{-1.0}, 1.0, -1.0, 2.0\}$$

A finite duration discrete time s/e that satisfies the condition $x(n) = 0$ for $n < 0$ is represented as,

$$x(n) = \{\underset{\uparrow}{-1.0}, 1.0, -1.0, 2.0\} \text{ (or) } x(n) = \{-1.0, 1.0, -1.0, 2.0\}$$

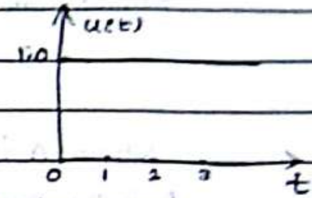
Elementary Continuous Time Signals:-

- ✓ Unit step function
- ✓ Unit Ramp function
- ✓ Unit Impulse function
- ✓ Exponential signal
- ✓ Sinusoidal signal

Unit step function:-

The unit step function is defined as

$$u(t) = 1 \text{ for } t \geq 0$$
$$= 0 \text{ for } t < 0$$



Here unit step means that the amplitude of $u(t)$ is equal to one.

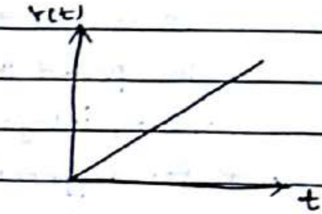
Unit Ramp function:-

The unit ramp function is defined as

$$r(t) = t \text{ for } t \geq 0$$
$$= 0 \text{ for } t < 0$$

(or)

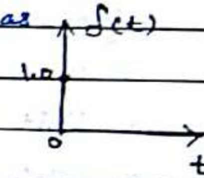
$$r(t) = t\,u(t)$$



Impulse function:-

The unit impulse function $\delta(t)$ is defined as

$$\delta(t) = 1;\ t = 0$$
$$= 0;\ t \neq 0$$

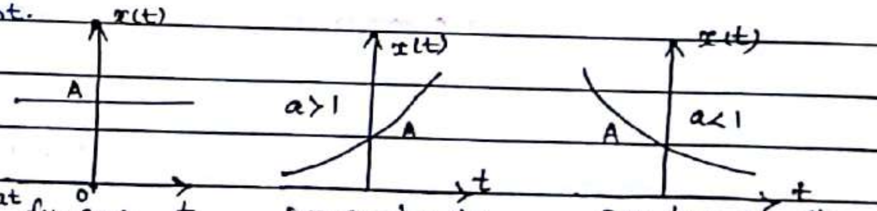


Exponential function:-

A real exponential signal is defined as

$$x(t) = Ae^{at}$$

Where both A and a real. Depending on the value of 'a' we get different signals. If a is +ve the s/e $x(t)$ is a growing exponential. If a is −ve, the s/e is a decaying exponential. For $a=0$, $x(t)$ is constant.



$x(t) = e^{at}$ for $a = 0$
DC signal.

Exp. growing s/e

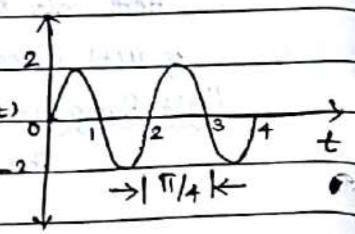
Exp. decaying s/e

Sinusoidal Signal:-

A Continuous - time sinusoidal s/e is given by

$$x(t) = A \sin(\Omega t + \theta)$$

Where A is amplitude, Ω is the freq in radians per second and θ is the phase angle in radians.



Properties of Analog Sinusoidal signal:-

✓ The s/e is periodic satisfying the Condition $x(t+T) = x(t)$

Where T is the fundamental period.

The period the continuous time sinusoidal signal can take any value, integral fraction or irrational.

✓ For different values of frequencies the continuous - time sinusoidal signals are themselves different.

Elementary Discrete - time signals:-

✓ Unit step sequence

✓ Unit ramp Sequence

✓ Unit impulse Sequence

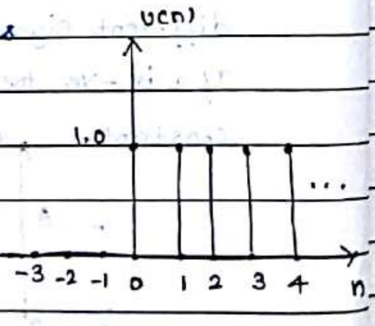
✓ Exponential Sequence

✓ Sinusoidal Signal

Unit Step Sequence:-

The unit step sequence is defined as

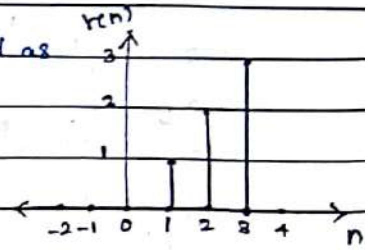
$$u(n) = 1 \text{ for } n \geq 0$$
$$= 0 \text{ for } n < 0$$



Unit ramp Sequence:-

The unit ramp sequence is defined as

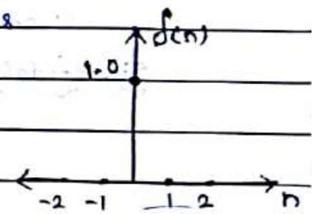
$$r(n) = n \text{ for } n \geq 0$$
$$= 0 \text{ for } n < 0$$



Unit Impulse Sequence:-

The unit impulse sequence is defined as

$$\delta(n) = 1 \text{ for } n = 0$$
$$= 0 \text{ for } n \neq 0$$



The unit impulse function has the following Properties

$$\delta(n) = u(n) - u(n-1)$$

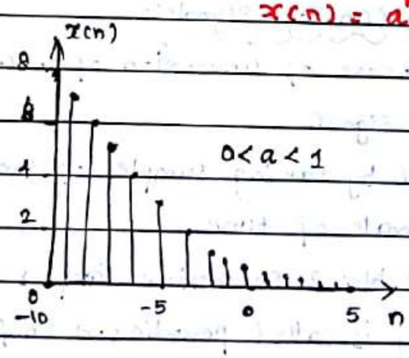
$$u(n) = \sum_{k=-\infty}^{n} \delta(k)$$

$$\sum_{n=-\infty}^{\infty} x(n)\delta(n-n_0) = x(n_0)$$

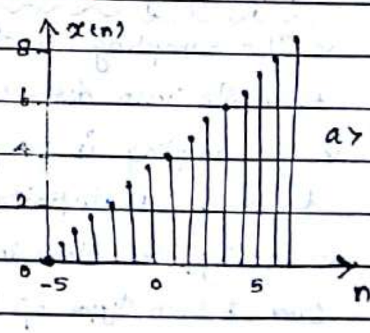
Exponential Sequence:-

The exponential s/e is a sequence of the form

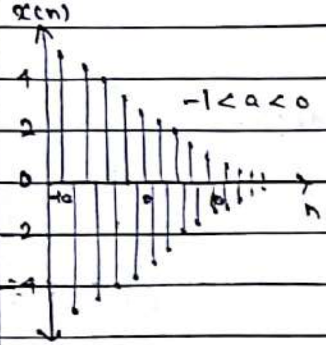
$$x(n) = a^n \text{ for all } n$$

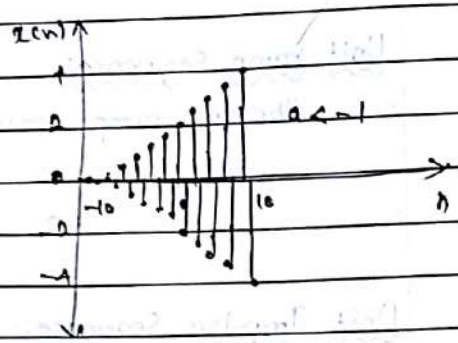


Sequence decays exponentially



Sequence grows exponentially





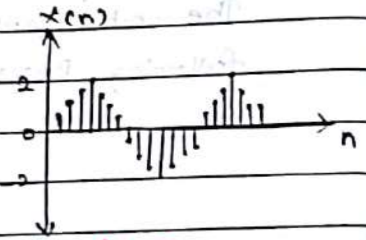
When $a < 0$, the discrete time exp. s/e takes alternating signs.

Sinusoidal Signal:-

The discrete time sinusoidal s/e is given by

$$x(n) = A\cos(\omega_0 n + \phi)$$

Where ω_0 is the freq. and ϕ is the phase.



Using Euler's identity, we can write

$$A\cos(\omega_0 n + \phi) = \frac{A}{2} e^{j\phi} e^{j\omega_0 n} + \frac{A}{2} e^{-j\phi} e^{-j\omega_0 n}$$

Since $|e^{j\omega_0 n}|^2 = 1$, the energy of the s/e is infinite and the avg. Pr. of the s/e is 1.

Sampling of Continuous Time (Analog) Signals:-

✓ The sampling is the process of conversion of a continuous time s/e into discrete time signal.

✓ The sampling is performed by taking samples of continuous time s/e at definite intervals of time.

✓ Usually, the time interval b/w 2 successive samples will be same & such type of sampling is called periodic (or) uniform sampling.

✓ The time interval b/w successive samples is called sampling time (or sampling period or interval) and it is denoted by "T".

✓ The unit of sampling period is seconds.

✓ The inverse of sampling period is called sampling frequency (or sampling rate) and it is denoted by "F_s". The unit is Hertz.

Let $x_a(t)$ = Analog / continuous time s/e

$x(n)$ = Discrete time s/e obtained by sampling $x_a(t)$

Mathematically, the relation b/w $x(n)$ and $x_a(t)$ can be expressed as,

$$x(n) = x_a(t)\Big|_{t=nT} = x_a(nT) = x_a\left(\frac{n}{F_s}\right), \quad -\infty < n < \infty$$

Where, T = Sampling period or interval in seconds

$F_s = \frac{1}{T}$ = Sampling rate or sampling freq. in Hz.

Sampling and Aliasing:-

While sampling analog s/e, the infinite freq. range continuous time s/e are mapped (or converted) to finite freq. range discrete time signals.

The relation b/w freq of analog and discrete time s/e is

$$f = \frac{F}{F_s} \quad \text{— (a)}$$

The range of freq of discrete time s/e is

$$-1/2 \le f \le 1/2 \quad \text{— (b)}$$

On substituting for f from eqn (a) in eqn (b) we get

$$-1/2 \le \frac{F}{F_s} \le 1/2 \quad \text{— (c)}$$

On multiplying eqn. ⓒ by F_s, we get

$$-\frac{F_s}{2} \le F \le \frac{F_s}{2} \quad \text{—— ⓓ}$$

The Phenomenon of high frequency Component getting the identity of low frequency component during sampling is called "Aliasing".

Let, F_{max} be max freq. of analog s/e that can be uniquely represented as discrete time s/e when sampled at a freq F_s.

Now, $F_{max} = \frac{F_s}{2}$

$\therefore F_s = 2F_{max}$

To avoid aliasing,

$F_s \ge 2F_{max}$

When Sampling Frequency F_s is equal to $2F_{max}$, the Sampling rate is called "Nyquist rate."

Sampling Theorem:-

A band limited continuous time s/e with highest frequency (bandwidth) F_m hertz can be uniquely recovered from its samples provided that the sampling rate F_s is greater than (or) equal to $2F_m$ Samples per second.

Note:- (Periodic Signal)

$x(N+n) = x(n)$ for all n. —— ①

$x(n) = A\sin(\omega_0 n + \theta)$ —— ②

A → Amplitude, ω_0 → freq, θ → Phase shift

units ω_0 → rad/samples θ → radians

A discrete time seq. is periodic if it satisfies the condition ① eqn. $x(n+N) = x(n)$ —— ③

From Eqn. ②

$x(n+N) = A\sin(\omega_0(n+N)+\theta)$

$= A\sin(\omega_0 n + \omega_0 N + \theta)$

For $x(n)$, Eqn. ③ satisfies if and only if $\omega_0 N$ is an integer multiple of 2π. i.e., $\omega_0 N = 2\pi m$ —— ④ for which $\omega_0 = 2\pi(m/N)$ —— ⑤ (or) $N = 2\pi(m/\omega_0)$ —— ⑥

Classification of Discrete Time Signals:

The discrete time signals are classified depending on their characteristics.

- ✓ Deterministic and nondeterministic signals.
- ✓ Periodic and aperiodic Signals
- ✓ Symmetric (Even) and Asymmetric (odd) signals
- ✓ Energy and Power signals.
- ✓ Causal and non causal Signals

Deterministic Signal:-

The signals that can be completely specified by mathematical equations are called deterministic s/e.

Eg:- step, Ramp, Exponential etc.,

Non- Deterministic Signal:-

The signals whose characteristics are random in nature are called Non- deterministic signals.

Eg:- Noise.

Periodic signal:-

When a discrete time s/e $x(n)$, satisfies the condition $x(n+N) = x(n)$ for integer values of N, then the discrete time s/e $x(n)$ is called Periodic Signal. Here N is the number of samples of a period.

i.e., if $x(n+N) = x(n)$ for all n, then $x(n)$ is periodic.

Fundamental Period:-

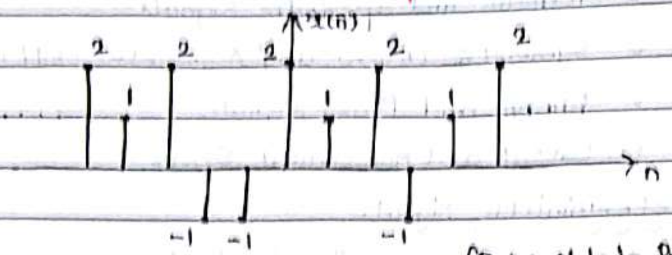
The smallest value of N for which the above equation is true is called fundamental period.

Aperiodic (or) Non-Periodic Signal:-

If there is no value of N

that satisfies the above equation, then $x(n)$ is called aperiodic (or) nonperiodic signal.

$$x(n) = \{ \ldots\ldots 2, 1, 2, -1, -1, \underset{\uparrow}{2}, 1, 2, -1, 1, 2 \ldots \}$$



(Refer Note in Previous page)

Problems:

Determine whether following signals are periodic or not. If periodic find the fundamental period.

(PRB) (1) $x(n) = e^{j6\pi n}$

$x(n) = e^{j6\pi n}$, $\omega_0 = 6\pi$

The fundamental frequency is multiple of π. Therefore, the s/e is periodic.

From Eqn. (1)

$$N = 2\pi\left(m/\omega_0\right) = 2\pi\left(m/6\pi\right)$$

The min. value of m for which N is integer of 3.

$$N = 2\pi\left(\frac{3}{6\pi}\right) = 1$$

Therefore, the fundamental period = 1

(PRB) (2) $x(n) = e^{j3/5(n+1/2)}$

$\omega_0 = 3/5$, which is not a multiple of π.

Therefore the s/e is aperiodic

(PRB) (3) $x(n) = \cos(2\pi/3)n$

$\omega_0 = \frac{2\pi}{3}$ The s/e is periodic.

The fundamental period, $N = 2\pi\left(\frac{m}{\omega_0}\right) = 2\pi\left(\frac{m}{2\pi/3}\right)$

$$N = 2\pi\left(\frac{3m}{2\pi}\right) = 3m$$

$N = 3$ (for $m = 1$)

$\therefore$ The fundamental period of the s/e is 3.

(PRB) (4) $x(n) = \cos(\pi/3 n) + \cos(3\pi/4)n$

The fundamental period of the s/e $\cos(\pi/3\, n)$ is

$$N_1 = 2\pi\left(\frac{m}{\omega_0}\right) = 2\pi\left(\frac{m}{\pi/3}\right) = 2\pi\left(\frac{3m}{\pi}\right) = 6 \quad (\because m=1)$$

$$N_2 = 2\pi\left(m/\omega_0\right) = 2\pi\left(\frac{m}{3\pi/4}\right) = 2\pi\left(\frac{4m}{3\pi}\right) = 8 \quad (\because m = 3)$$

$$\frac{N_1}{N_2} = \frac{6}{8} = \frac{3}{4}$$

$\Rightarrow N = 4N_1 = 3N_2 = 24 \qquad \therefore N = 24$

HW (PRB) (5) $x(n) = \sin\left(\frac{\pi n}{4}\right)$ = 8 samples ($\because N = 8$) - Periodic

(6) $x(n) = e^{j2n}$ - apperiodic

$\omega_0 = 2$, which is not a multiple of π. Therefore the s/e is apperiodic.

(7) $x(n) = \cos\frac{\pi}{4}n + \cos 2n$ - apperiodic

s/e is apperiodic

Symmetric (Even) and Asymmetric (odd) Signals :-

The discrete time signals may exhibit Symmetry (or) antisymmetry with respect to $n=0$.

Even Signal:-

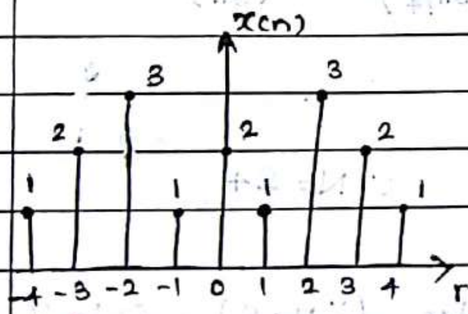
When a discrete time s/e (DTS) exhibits Symmetry with respect to $n=0$ then It is called an even signal. Therefore the even s/e satisfies the Condition,

$$x(-n) = x(n) \quad \text{—— (a)}$$

odd Signal:-

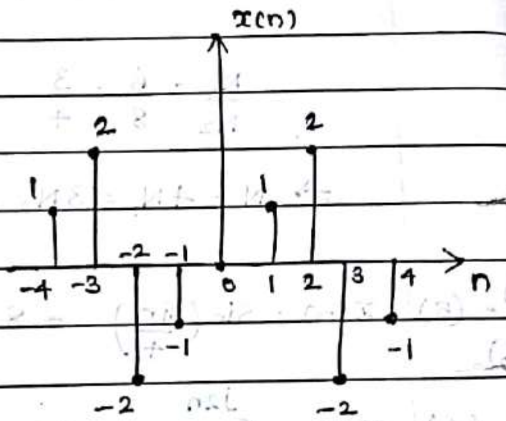
When a DTS exhibits asymmetry with respect to $n=0$, then it is called an odd signal. Therefore the odd s/e Satisfies the condition,

$$x(-n) = -x(n) \quad \text{—— (b)}$$



$x(n) = \{1, 2, 3, 1, \underset{\uparrow}{2}, 1, 3, 2, 1\}$

Symmetric (or) even Signal



$x(n) = \{1, 2, -2, -1, \underset{\uparrow}{0}, 1, 2, -2, -1\}$

Asymmetric (or) odd Signal

A DTS $x(n)$ which is neither even nor odd can be expressed as a sum of even and odd signal

let, $x(n) = x_e(n) + x_o(n)$

Where, $x_e(n)$ = Even part of $x(n)$

$x_o(n)$ = odd part of $x(n)$

Note:- If $x(n)$ is even then its odd part will be zero.
If $x(n)$ is odd then its even part will be zero.

Now it can be proved that,

Even part, $x_e(n) = 1/2[x(n) + x(-n)]$

odd Part, $x_o(n) = 1/2[x(n) - x(-n)]$

Proof:-

let, $x(n) = x_e(n) + x_o(n) \quad \text{—— (1)}$

on replacing n by $-n$ in equation (1), we get

$$x(-n) = x_e(-n) + x_o(-n) \quad \text{—— (2)}$$

from eqn. (a) & (b) Since $x_e(n)$ is even, $x_e(-n) = x_e(n)$

Since $x_o(n)$ is odd, $x_o(-n) = -x_o(n)$

Hence the eqn (2) can be written as

$$x(-n) = x_e(n) - x_o(n) \quad \text{—— (3)}$$

on adding eqn. (1) & (3) we get,

$$x(n) + x(-n) = 2x_e(n)$$

$$\therefore x_e(n) = \frac{1}{2}[x(n) + x(-n)]$$

on Subtracting eqn (3) from eqn. (1) we get,

$$x(n) - x(-n) = 2x_o(n)$$

$$\therefore x_o(n) = \frac{1}{2}[x(n) - x(-n)]$$

Hence proved.

Energy and Power signals:

If energy (E) of a DTS is finite & Non zero, then the DTS is called an energy s/e. The exponential s/e's are examples of energy signals.

The energy of a s/e may be finite or infinite, and can be applied to complex valued & real valued s/e.

The energy E of a DTS x(n) is defined as

$$\text{Energy}, E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

If Power (P) of a DTS is finite and non zero, then the DTS is called a power signal. The periodic s/e are examples of Power signals.

The average Power of a DTS x(n) is defined as

$$\text{Power}, P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

For energy s/e, the energy will be finite and avg. Pr. will be zero.
For Pr. s/e, the avg. Pr. is finite and energy will be infinite.

∴ For Energy s/e, $0 < E < \infty$ and $P = 0$

For Power s/e, $0 < P < \infty$ and $E = \infty$

Problem:-

Determine whether the following signals are energy (or) Power signals.

(1) (PRB) $x(n) = (1/3)^n u(n)$

$x(n) = (1/3)^n u(n)$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

⑪

$$= \sum_{n=0}^{\infty} \left[(1/3)^n\right]^2$$

unit step seq.
$\because u(n) = 1$ for $n \geq 0$
$= 0$ for $n < 0$

$$= \sum_{n=0}^{\infty} (1/9)^n$$

Infinite geometric series sum formulae
$\left[\sum_{n=0}^{\infty} c^n = \frac{1}{1-c}\right]$

$$= \left(\frac{1}{9}\right)^0 + (1/9)^1 + (1/9)^2 + \dots \infty$$

$$E = \frac{1}{1-\frac{1}{9}} = \frac{9}{8} \text{ joules}$$
$\left[\because 1 + a + a^2 + a^3 + \dots + \infty = \frac{1}{1-a}\right]$

According to the Power formulae

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} \left(\frac{1}{9}\right)^n$$

Infinite geometric series sum formulae
$\left[\sum_{n=0}^{N} c^n = \frac{1-c^{N+1}}{1-c}\right]$

$$\therefore P = \lim_{N\to\infty} \frac{1}{2N+1} \left[\frac{1-(1/9)^{N+1}}{1-\frac{1}{9}}\right] = 0$$

The energy is finite and Power is zero. Therefore the s/e is an energy signal.

(2) (ANK) tutorial

$x(n) = (1/4)^n u(n)$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=0}^{\infty} |(1/4)^n|^2 = 1.067 \text{ joules}$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} |\underbrace{(1/4)^n}_{0.25}|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} (0.0625)^n$$

(5) $x(n) = e^{j(\pi/2 n + \pi/4)}$
(PRB)

$$x(n) = e^{j\left(\frac{\pi}{2}n + \frac{\pi}{4}\right)}$$

HW

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 \qquad \left[\because \left|e^{j(\omega+\theta)}\right| = 1\right]$$

$$= \sum_{n=-\infty}^{\infty} \left|e^{j(\pi/2 n + \pi/4)}\right|^2$$

$$E = \sum_{n=-\infty}^{\infty} 1^n = \infty \text{ (Infinite)}$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left|e^{j\left(\frac{\pi}{2}n + \pi/4\right)}\right|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} 1 \qquad \left[\because \sum_{n=-N}^{N} 1 = 2N+1\right]$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} (2N+1) = 1\,W \text{ (finite)}$$

So the given S/e is Power Signal.

HW

(6) $x(n) = \sin(\pi/4\, n)$ $E = \infty$, $P = 1/2\,W$
(PRB)

Power Signal.

(7) $x(n) = e^{2n} u(n)$ $E = \infty$, $P = \infty$
(PRB)

Signal is neither Power nor energy Signal.

(8) $x(n) = \cos(\omega_0 n)\, u(n)$
(PRB)

Power Signal.

(9) $x(n) = u(n+2) - u(n-2)$
(PRB)

Energy Signal.

Causal, Non-Causal and Anticausal Signals:-

Causal Signal:-

A DTS is said to be causal, if it is defined for $n \geq 0$. Therefore if $x(n)$ is causal, then $x(n) = 0$ for $n < 0$.

Eg:-

$$x_1(n) = a^n u(n)$$

$$x_2(n) = \{\underset{\uparrow}{1}, 2, -3, -1, 2\}$$

Non Causal Signal:-

A DTS is said to be non causal, if it is defined for either $n \leq 0$, (or) for both $n \leq 0$ & $n > 0$. Therefore if $x(n)$ is non causal, then $x(n) \neq 0$ for $n < 0$. A non causal s/e can be converted to causal s/e by multiplying the non causal s/e by a unit step s/e $u(n)$.

Eg:-

$$x_3(n) = a^n u(-n+1)$$

$$x_4(n) = \{1, -2, \underset{\uparrow}{-3}, -1, 2\}$$

Anticausal Signal:-

When a non causal DTS is defined only for $n \leq 0$, it is called an anticausal signal.

Eg:-

$$x_5(n) = \{1, 2, -3, -2, -1, 1, \underset{\uparrow}{3}\}$$

$$x_6(n) = \{\ldots\ldots\ldots -2, -4, 1, 3, \underset{\uparrow}{2}\}$$

Operations on Discrete Time Signals:-

Signal processing is a group of basic operations applied an input s/e resulting in another s/e as the o/p. The mathematical transformation from one s/e to another is represented as

$$y(n) = T[x(n)]$$

The basic set of operations are

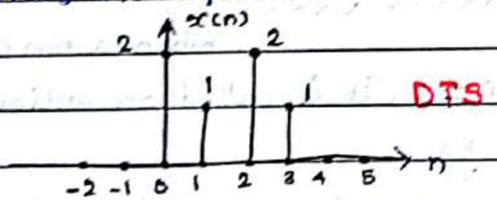
- ✓ Shifting
- ✓ Time reversal
- ✓ Time Scaling
- ✓ Scalar multiplication
- ✓ Signal multiplier
- ✓ Signal addition

Shifting:-

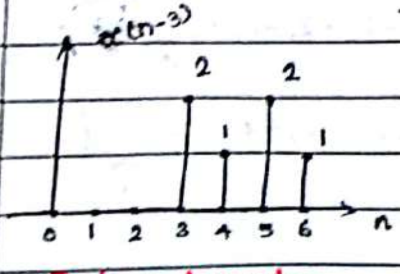
The shift operation takes the input sequence & shifts the values by an integer increment of the independent variable. The shifting may delay (or) advance the sequences in time. It is represented as,

$$y(n) = x(n-k)$$

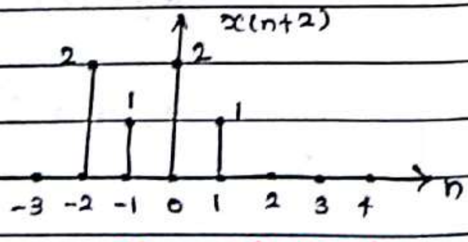
where, $x(n)$ is the input and $y(n)$ is the output. If k is positive, the shifting delays the sequence. If k is negative, the shifting advances the sequence.



DTS



Delayed version

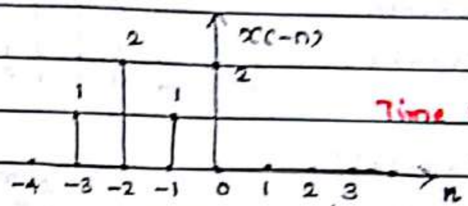


Advanced version

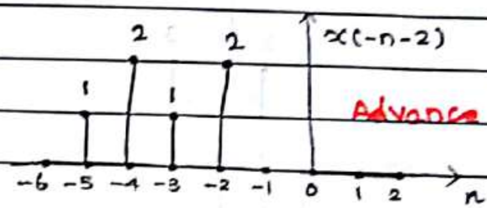
Time Reversal:-

The time reversal of sequence $x(n)$ can be obtained by folding the sequence about $n=0$. It is denoted as $x(-n)$.

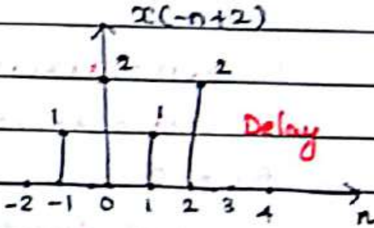
The signal $x(-n+2)$ is $x(-n)$ delayed by two units of time and $x(-n-2)$ is $x(-n)$ advanced by 2 units of time.



Time reversal



Advance



Delay

Time Scaling:-

This is accomplished by replacing n by λn in the sequence x(n). There are two ways of time scaling a DTS. They are down sampling and up sampling.

In a signal x(n), if n is replaced by Dn, where D is an integer, then it is called "Down sampling" (compression)

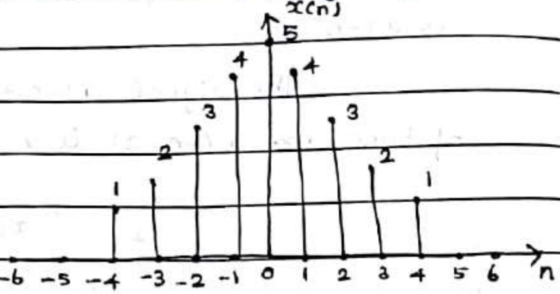
In a signal x(n), if n is replaced by $\frac{n}{I}$, where I is an integer, then it is called "up sampling". (Expansion)

Down Sampling, $y(n) = x(\lambda n)$

Up Sampling, $y(n) = x(\frac{n}{\lambda})$

Eg:- $y(n) = x(2n)$

We can plot the sequence $y(n)$ by substituting different values for n.



For $n=-1$

$y(-1) = x(2(-1))$

$y(-1) = x(-2) = 3$

For $n=-2$

$y(-2) = x(2(-2))$

$y(-2) = x(-4) = 1$

For $n=1$

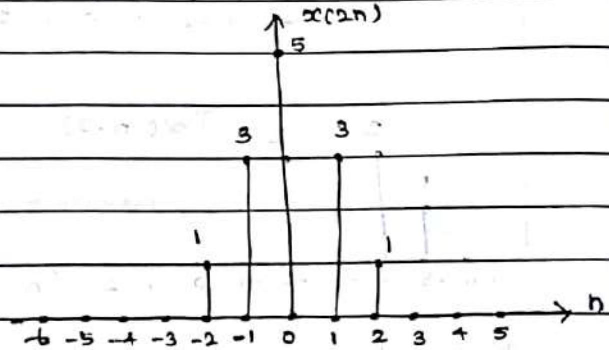
$y(1) = x(2(1))$

$y(1) = x(2) = 3$

For $n=2$

$y(2) = x(2(2))$

$y(2) = x(4) = 1$



Eg:- $y'(n) = x(n/2)$

For $n=1$

$y'(1) = x(1/2)$

For $n=2$

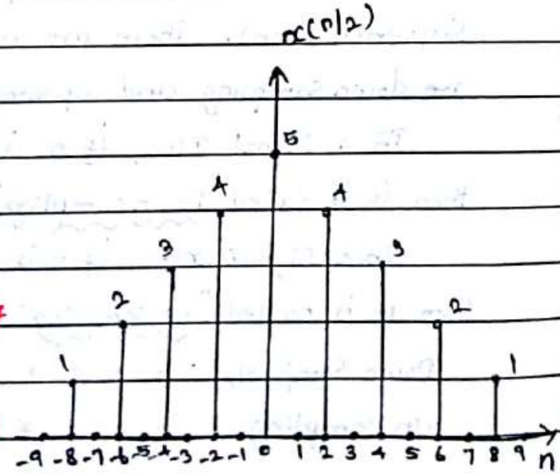
$y'(2) = x(2/2) = x(1) = 4$

For $n=-2$

$y'(-2) = x(-2/2) = x(-1) = 4$

For $n=4$

$y'(4) = x(4/2) = x(2) = 3$

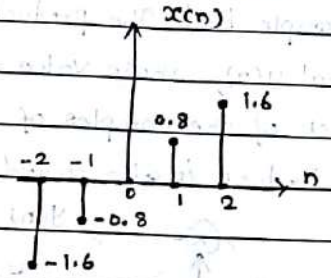


Folding:-

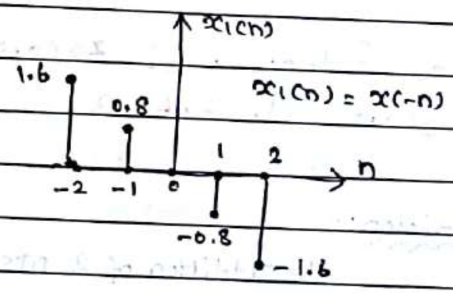
The folding of a DTS $x(n)$ is performed by changing the sign of the time base n in $x(n)$. The folding operation produces a s/e $x(-n)$ which is a mirror image of the s/e $x(n)$ with respect to time origin $n=0$. It is also called as "Reflection" (or) "Transpose" of Discrete Time s/e.

Eg:-

Let $x(n) = 0.8\,n$; $-2 \le n \le 2$



Now the folded s/e, $x_1(n) = x(-n) = -0.8(n)$; $-2 \le n \le 2$



Multiplication:-

There are 2 approaches of multiplication in DTS. They are

- ✓ Scalar multiplication
- ✓ Signal multiplier

Scalar Multiplication:-

The signal x(n) is multiplied by a scale factor a.

$$x(n) \xrightarrow{a} y(n) = a\,x(n)$$

Eg:-

$x(n) = \{1, -2, 1, 2\}$ & $a = 2$

Then the signal y(n) will be $y(n) = \{2, -4, 2, 4\}$

Signal Multiplier:-

The multiplication of 2 DTS is performed on a sample-by-sample basis. The product of 2 signals $x_1(n)$ and $x_2(n)$ is a signal y(n), whose value at any instant is equal to the product of the samples of these 2 signals at that instant. The product is also called modulation.

$$x_1(n) \rightarrow \otimes \rightarrow y(n), \quad x_2(n) \uparrow$$

$$y(n) = x_1(n)\,x_2(n)$$

Eg:-

$x_1(n) = \{-1, 2, -3, -2\}$ $x_2(n) = \{1, -1, -2, 1\}$

then $y(n) = \{-1, -2, 6, -2\}$

Signal Addition:-

The addition of 2 DTS is performed on a sample-by-sample basis. The sum of 2 s/e $x_1(n)$ and $x_2(n)$ is a signal y(n), whose value at any instant is equal to the sum of the samples of these 2 s/e at that instant.

$$y(n) = x_1(n) + x_2(n)$$

$$x_1(n) \rightarrow \oplus \rightarrow y(n), \quad x_2(n) \uparrow$$

Eg:-

$x_1(n) = \{2, 2, -1\}$ $x_2(n) = \{-1, 1, 2\}$

$y(n) = \{1, 3, 1\}$

Discrete Time System:-

A DTS/m is a device or an algorithm that operates on a discrete time s/e x(n), according to some well defined rule, to produce another DTS y(n) called the o/p s/e. The relationship between x(n) and y(n) is

$$y(n) = T[x(n)]$$

$$\underset{x(n)}{\text{I/p s/e}} \rightarrow \boxed{\text{DTS/m}} \rightarrow \underset{\text{o/p s/e}}{y(n)}$$

Classification of DTS/m:-

The DTS/m are classified according to their general properties and characteristics. They are

- ✓ Static and dynamic systems.
- ✓ Causal and Noncausal System
- ✓ Linear and Non-Linear system
- ✓ Time Variant and Time invariant system
- ✓ stable and unstable systems
- ✓ FIR and IIR Systems
- ✓ Recursive and Nonrecursive systems.

Static and dynamic System:-

A DTS/m is called "static" (or) "memoryless" s/m if its o/p at any instant n depends at most on the I/p sample at the same time but not on the past (or) future samples of the input. In any other case, the s/m is said to be "dynamic" (or) to have memory.

Summary of analysis and synthesis of eqn for FT and DFT:-

Consider $\tilde{x}(n)$ to be a periodic sequence of time period 'N'. If the time period 'N' tends to ∞ then the periodic sequence $\tilde{x}(n)$ will become a aperiodic sequence $x(n)$.

$$\therefore x(n) = \lim_{N\to\infty} \tilde{x}(n)$$

Let c_k be the Fourier coeff,

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} \tilde{x}(n) e^{-\frac{j2\pi kn}{N}}$$

$$\Rightarrow N c_k = \sum_{n=0}^{N-1} \tilde{x}(n) e^{-\frac{j2\pi kn}{N}}$$

Since $\tilde{x}(n)$ is periodic, for even values of N the summation index in the above eqn can be modified as $n = -\left(\frac{N}{2}-1\right)$ to $+\frac{N}{2}$ and for odd values of N the summation index will be $n = -\frac{N}{2}$ to $\frac{N}{2}$.

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For even values of 'N',

$$N c_k = \sum_{n=-\left(\frac{N}{2}-1\right)}^{N/2} \tilde{x}(n) e^{\frac{-j2\pi kn}{N}}$$

$$\therefore N c_k = \sum_{n=-\left(\frac{N}{2}-1\right)}^{N/2} \tilde{x}(n) e^{-j\omega_k n} \rightarrow ①$$

where $\omega_k = \frac{2\pi k}{N}$

Now $N c_k$ is defined as a function of $e^{j\omega_k}$

$$\therefore X\left(e^{j\omega_k}\right) = N c_k \rightarrow ②$$

Sub ① in ②

$$\Rightarrow X\left(e^{j\omega_k}\right) = \sum_{n=-\left(\frac{N}{2}-1\right)}^{N/2} \tilde{x}(n) e^{-j\omega_k n} \rightarrow ③$$

In ③ when $N \rightarrow \infty$, $\tilde{x}(n)$ becomes $x(n)$, ω_k becomes ω and the summation index becomes $-\infty$ to ∞

$$\therefore X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \rightarrow ④$$

Eqn ④ represents Fourier Transform of $x(n)$ which represents the non-periodic discrete time signal in frequency domain.

Consider the Fourier series representation of $\tilde{x}(n)$,

$$\tilde{x}(n) = \sum_{k=0}^{N-1} c_k e^{\frac{j2\pi kn}{N}}$$

Multiplying and dividing by $\frac{N}{2\pi}$

$$\Rightarrow \tilde{x}(n) = \frac{N}{2\pi} \times \frac{2\pi}{N} \sum_{k=0}^{N-1} c_k e^{\frac{j2\pi kn}{N}}$$

$$\Rightarrow \tilde{x}(n) = \frac{N}{2\pi} \sum_{k=0}^{N-1} c_k e^{\frac{j2\pi kn}{N}} \cdot \frac{2\pi}{N} \qquad \left[\because \omega_k = \frac{2\pi k}{N}\right]$$

$$\tilde{x}(n) = \frac{1}{2\pi} \sum_{k=0}^{N-1} X(e^{j\omega k})(e^{j\omega_k n}) \cdot \frac{2\pi}{N} \rightarrow (5)$$

In (5); when $N \to \infty$

$\Rightarrow \tilde{x}(n) \to x(n)$; $\omega_k \to \omega$; $\frac{2\pi}{N} \to d\omega$ and the summation becomes integral with limits 0 and 2π.

$$\therefore x(n) = \frac{1}{2\pi} \int_0^{2\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega \rightarrow (6)$$

This is the IFT of $x(n)$ which is used to extract DT signal from the frequency domain representation.

Since eqn (4) extracts the frequency components of DT signal, the transmission using (4) is called as the analysis of DT signal $x(n)$.

Since eqn (6) integrates or combines the frequency components of DT signal, the inverse transformation using (6) is called synthesis of DT signal $x(n)$.

Frequency domain sampling:

$$X(t) \xrightarrow{FT} X(e^{j\omega})$$

Aperiodic CT signal

$$x(n) \xrightarrow{FT} X(\omega)$$

Aperiodic DT signal

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

W.k.t., $\omega = \frac{2\pi k}{N}$

$$X\left(\frac{2\pi k}{N}\right) = \sum_{n=-\infty}^{\infty} x(n) e^{\frac{-j2\pi k n}{N}} \quad ; \; k = 0, 1, \ldots\ldots, N-1$$

The summation can be subdivided into

$$X\left(\frac{2\pi k}{N}\right) = \ldots\ldots + \sum_{n=-N}^{-1} x(n) e^{\frac{-j2\pi k n}{N}} + \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi k n}{N}} +$$

$$\sum_{n=N}^{2N-1} x(n) e^{\frac{-j2\pi k n}{N}}$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n) e^{\frac{-j2\pi k n}{N}}$$

Changing the index of the inner summation from n to n-lN.

$$\therefore X\left(\frac{2\pi k}{N}\right) = \sum_{l=-\infty}^{\infty} \sum_{n=0}^{N-1} x(n-lN) e^{\frac{-j2\pi k n}{N}}$$

Changing the order of summation

$$\therefore X\left(\frac{2\pi k}{N}\right) = \sum_{n=0}^{N-1} \left[\sum_{l=-\infty}^{\infty} x(n-lN)\right] e^{\frac{-j2\pi k n}{N}}$$

$$\therefore X\left(\frac{2\pi k}{N}\right) = \sum_{n=0}^{N-1} x_p(n) \; e^{\frac{-j2\pi k n}{N}} \longrightarrow (1)$$

The signal $x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$

where $x_p(n)$ is periodic repetion of $x(n)$ for every N samples.

Expanding $x_p(n)$ in fourier series,

$$\therefore x_p(n) = \sum_{k=0}^{N-1} c_k e^{\frac{j2\pi k n}{N}} \quad ; n = 0, 1, 2, \ldots\ldots, N-1$$

The fourier coeff,

$$c_k = \frac{1}{N} \sum_{k=0}^{N-1} x_p(n) e^{-\frac{j2\pi kn}{N}} \quad ; k=0,1,\ldots,N-1 \rightarrow (2)$$

Comparing (1) and (2)

$$c_k = \frac{1}{N} x\left(\frac{2\pi k}{N}\right) \quad ; \quad k=0,1,\ldots,N-1$$

Sub c_k in $x_p(n)$,

$$x_p(n) = \frac{1}{N} \sum_{k=0}^{N-1} x\left(\frac{2\pi k}{N}\right) e^{\frac{j2\pi kn}{N}} \quad ; n=0,1,\ldots,N-1$$

where $x_p(n)$ is reconstruction. But original signal is $x(n)$. Thus a relationship must be developed between $x(n)$ and $x_p(n)$.

$$x(n) = x_p(n) \quad ; \quad 0 \leq n \leq N-1$$

where L is length of $x(n)$

$L \leq N \rightarrow$ period of $x_p(n)$

Thus $x(n) = x_p(n) \quad ; \quad 0 \leq n \leq L$

$L \leq n \leq N-1$

$$\therefore x(n) = \begin{cases} x_p(n) & ; \quad 0 \leq n \leq N-1 \\ 0 & ; \quad \text{else} \end{cases}$$

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UNIT-2

Frequency Transformations

Introduction to DFT:-

Discrete Fourier Transform:- (DFT)

It is a powerful computation tool which allows us to evaluate the FT $X(e^{j\omega})$ on a digital computer (or) specially designed hardware. Since $X(e^{j\omega})$ is continuous and periodic, DFT is obtained by sampling one period of the FT at a finite no. of freq. points.

It plays an important role in the implementation of many S/e processing algorithms. It is used to perform linear filtering operations in the freq. domain.

DFT of a sequence is periodic, and we are interested in freq. range 0 to 2π. There are infinitely many ω in this range. To compute N equally spaced pts. over the interval $0 \le \omega < 2\pi$, then the N points should be located at

$$\omega_k = \frac{2\pi}{N} k \quad k = 0, 1, 2, \dots N-1$$

These N equally spaced freq. samples of the DTFT are known as DFT denoted by $X(k)$ is

$$X(k) = X(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N} k} \quad 0 \le k \le N-1$$

Definition:-

Let $x(n)$ = DTS of length L

$X(k)$ = DFT of $x(n)$

Now, the N-point DFT of $x(n)$, where $N \ge L$ is defined as,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \; ; \text{for } k = 0, 1, 2, \dots N-1$$

$$X(k) = \{X(0), X(1), X(2), \dots X(N-1)\}$$

Inverse DFT:-

Let $x(n)$ = DTS

$X(k)$ = N-point DFT of $x(n)$

The inverse DFT of the seq. $X(k)$ of length N is defined as,

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi kn}{N}} \; ; \text{for } n = 0, 1, 2, \dots N-1$$

$$x(n) \underset{\text{IDFT}}{\overset{\text{DFT}}{\rightleftharpoons}} X(k)$$

Problem:-

(PRB) (1) Find the DFT of a sequence $x(n) = \{1, 1, 0, 0\}$

$x(n) = \{1, 1, 0, 0\}$

Let N = L = 4

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \; ; \; k = 0, 1, 2 \dots N-1$$

$$X(k) = \sum_{n=0}^{4-1} x(n)\, e^{\frac{-j2\pi kn}{4}} \; ; \; k = 0, 1, 2, 3$$

For K = 0, $X(0) = \sum_{n=0}^{3} x(n) = x(0) + x(1) + x(2) + x(3)$

$X(0) = 1 + 1 + 0 + 0 = 2$

For K = 1, $X(1) = \sum_{n=0}^{3} x(n)\, e^{\frac{-j\cancel{2}\pi n}{\cancel{4}}} = \sum_{n=0}^{3} x(n)\, e^{-j\pi n/2}$

$= x(0) + x(1)e^{-j\pi/2} + x(2)e^{-j\pi} + x(3)e^{-j3\pi/2}$

$X(1) = 1 + \cos\pi/2 - j\sin\pi/2 = 1 - j$

For k=2,

$$X(2) = \sum_{n=0}^{3} x(n) e^{-j\pi n}$$

$$= x(0) + x(1) e^{-j\pi} + x(2) e^{-j2\pi} + x(3) e^{-j3\pi}$$

$$X(2) = 1 + \cos\pi - j\sin\pi = 1-1 = 0$$

For k=3,

$$X(3) = \sum_{n=0}^{3} x(n) e^{-j3n\pi/2}$$

$$= x(0) + x(1) e^{-3j\pi/2} + x(2) e^{-j3\pi} + x(3) e^{-j9\pi/2}$$

$$X(3) = 1 + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} = 1+j$$

$$X(k) = \{2,\ 1-j,\ 0,\ 1+j\}$$

Tutorial

(2) $x(n) = \{0, 1, 2, 1\}$ $\quad X(k) = \{4, -2, 0, -2\}$ (ANK)

(3) $x(n) = \{0, 2, 4, 6\}$ $\quad X(k) = \{12, -4+j4, -4, -4-j4\}$ (Apr'08)

Properties of DFT:-

Linearity :-

It states that the DFT of a linear weighted combination of two or more s/s is equal to similar weighted combination of the DFT of individual signals.

Let, DFT $\{x_1(n)\} = X_1(k)$ & DFT $\{x_2(n)\} = X_2(k)$. Then by linearity property,

DFT $\{a_1 x_1(n) + a_2 x_2(n)\} = a_1 X_1(k) + a_2 X_2(k)$, where a_1 & a_2 constants.

Proof:-

$$X_1(k) = \text{DFT}\{x_1(n)\} = \sum_{n=0}^{N-1} x_1(n) e^{\frac{-j2\pi kn}{N}}$$

$$X_2(k) = \text{DFT}\{x_2(n)\} = \sum_{n=0}^{N-1} x_2(n) e^{\frac{-j2\pi kn}{N}}$$

$$\text{DFT}\{a_1 x_1(n) + a_2 x_2(n)\} = \sum_{n=0}^{N-1} [a_1 x_1(n) + a_2 x_2(n)] e^{\frac{-j2\pi kn}{N}}$$

$$= \sum_{n=0}^{N-1} [a_1 x_1(n) e^{\frac{-j2\pi kn}{N}} + a_2 x_2(n) e^{\frac{-j2\pi kn}{N}}]$$

$$= \sum_{n=0}^{N-1} a_1 x_1(n) e^{\frac{-j2\pi kn}{N}} + \sum_{n=0}^{N-1} a_2 x_2(n) e^{\frac{-j2\pi kn}{N}}$$

$$\text{DFT}\{a_1 x_1(n) + a_2 x_2(n)\} = a_1 X_1(k) + a_2 X_2(k)$$

Periodicity:-

If a seq. $x(n)$ is periodic with periodicity of N samples then N-point DFT, $X(k)$ is also periodic with a periodicity of N samples.

Hence if $x(n)$ & $X(k)$ are N-point DFT pair then,

$x(n+N) = x(n)$; for all n

$X(k+N) = X(k)$; for all k

Proof:-

$$X(k+N) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi n(k+N)}{N}}$$

$$= \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi nk}{N}} \overset{1}{\cancel{e^{\frac{-j2\pi nN}{N}}}}$$

[∵ for integer n, $e^{-j2\pi n} = 1$]

$$X(k+N) = X(k)$$

Symmetry Property

If the samples $x(n)$ are real, then extracting in freq. domain $X(0), \ldots, X(N-1)$ seems counter intutive, becoz from N bits of info. in one domain (time), we are deriving 2N bits of info in freq domain. This suggests that there is some redundancy in computation of $X(0) \ldots X(N-1)$. As per DFT property of symmetry

$$X(N-k) = X^*(k), \quad k = 0, 1, \ldots, N-1 \text{; where } * \rightarrow \text{ complex conjugate}$$

Proof :-

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}$$

$$X(N-k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi n(N-k)}{N}}$$

$\left(\because e^{-j2\pi n} = 1\right)$ n is an integer

$$= \sum_{n=0}^{N-1} x(n)\, \underbrace{e^{\frac{-j2\pi n \cancel{N}}{\cancel{N}}}}_{1} \cdot e^{\frac{j2\pi nk}{N}}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{\frac{j2\pi nk}{N}}$$

$\left(\because e^{\frac{j2\pi nk}{N}} \text{ is complex conjugate of } e^{-\frac{j2\pi nk}{N}}\right)$

$$X(N-k) = X^*(k)$$

Circular time Shift:-

It says that if a DTS is circularly shifted in time by m units then its DFT is multiplied by $e^{\frac{-j2\pi km}{N}}$.

i.e., $DFT\{x(n)\} = X(k)$, then $DFT\{x((n-m))_N\} = X(k)e^{\frac{-j2\pi km}{N}}$

Proof:-

$$DFT\{x((n-m))_N\} = \sum_{n=0}^{N-1} x((n-m))_N \, e^{\frac{-j2\pi kn}{N}}$$

let, $P = n-m$; $\therefore n = P+m$

$$= \sum_{P=0}^{N-1} x(p)\, e^{\frac{-j2\pi k(P+m)}{N}}$$

$$= \sum_{P=0}^{N-1} x(p)\, e^{\frac{-j2\pi kp}{N}} \, e^{\frac{-j2\pi km}{N}}$$

$$DFT\{x((n-m))_N\} = X(k)\, e^{\frac{-j2\pi km}{N}}$$

Time Reversal:-

It states that reversing the N-pt. Seq. in time is equivalent to reversing the DFT Sequence.

i.e, $DFT\{x(n)\} = X(k)$, then $DFT\{x(N-n)\} = X(N-k)$

Proof:-

$$DFT\{x(N-n)\} = \sum_{n=0}^{N-1} x(N-n)\, e^{\frac{-j2\pi kn}{N}} \quad [\text{let}, m = N-n, \therefore n = N-m]$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{\frac{-j2\pi k(N-m)}{N}}$$

Since,

[k is an integer, $e^{-j2\pi k} = 1$

m is an integer, $e^{-j2\pi m} = 1$]

$$= \sum_{m=0}^{N-1} x(m)\, \cancel{e^{\frac{-j2\pi kN}{N}}}\, e^{\frac{j2\pi km}{N}}$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{\frac{j2\pi km}{N}} \cdot \underbrace{e^{-j2\pi m}}_{1}$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{\frac{j2\pi km}{N}} \cdot \underbrace{e^{\frac{-j2\pi mN}{N}}}_{1}$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{\frac{-j2\pi m(N-k)}{N}}$$

$$DFT\{x(N-n)\} = X(N-k)$$

Conjugation:-

Let $x(n)$ be a complex N-Point discrete Seq. and $x^*(n)$ be its Conjugate Sequence.

Now if, $DFT\{x(n)\} = X(k)$, then $DFT\{x^*(n)\} = X^*(N-k)$

Proof:-

$$DFT\{x^*(n)\} = \sum_{n=0}^{N-1} x^*(n)\, e^{\frac{-j2\pi kn}{N}}$$

$$= \left[\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}\right]^*$$

$$= \left[\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \cdot \underbrace{e^{-j2\pi n}}_{1}\right]^* \quad [\because e^{-j2\pi n} = 1]$$

$$= \left[\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \cdot \underbrace{e^{\frac{-j2\pi nN}{N}}}_{1}\right]^*$$

$$= \left[\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi n(N-k)}{N}}\right]^*$$

$$= [X(N-k)]^*$$

$$DFT\{x^*(n)\} = X^*(N-k)$$

Circular frequency shift :-

It states that if a discrete time s/p is multiplied by $e^{\frac{j2\pi mn}{N}}$ its DFT is ⊙ly shifted by m units.

i.e. $DFT\{x(n)\} = X(k)$ then $DFT\{x(n)\, e^{\frac{j2\pi mn}{N}}\} = X((k-m))_N$

Proof :-

$$DFT\{x(n)\, e^{\frac{j2\pi mn}{N}}\} = \sum_{n=0}^{N-1} x(n)\, e^{\frac{j2\pi mn}{N}}\, e^{\frac{-j2\pi kn}{N}}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi(k-m)n}{N}}$$

$$DFT\{x(n)\, e^{\frac{j2\pi mn}{N}}\} = X((k-m))_N$$

Multiplication :-

It states that the DFT of product of 2 DT seq. is equivalent to ⊙lar convolution of the DFTs of the individual seq. scaled by a factor 1/N.

i.e. $DFT\{x(n)\} = X(k)$, then $DFT\{x_1(n)\, x_2(n)\} = \frac{1}{N}[X_1(k) \circledast X_2(k)]$

Proof :-

By definition of IDFT,

$$x_1(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_1(k)\, e^{\frac{j2\pi kn}{N}}$$

let k = m

$$= \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)\, e^{\frac{j2\pi mn}{N}} \quad (1)$$

By definition of DFT,

$$DFT\{x_1(n)\, x_2(n)\} = \sum_{n=0}^{N-1} x_1(n)\, x_2(n)\, e^{\frac{-j2\pi kn}{N}}$$

Sub. eqn (1) in $x_1(n)$, we get

$$= \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{m=0}^{N-1} X_1(m)\, e^{\frac{j2\pi mn}{N}}\right] x_2(n)\, e^{\frac{-j2\pi kn}{N}}$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} x_1(m) \left[\sum_{n=0}^{N-1} x_2(n) e^{\frac{-j2\pi kn}{N}} e^{\frac{j2\pi mn}{N}} \right]$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} x_1(m) \left[\sum_{n=0}^{N-1} x_2(n) e^{\frac{-j2\pi(k-m)n}{N}} \right]$$

[∵ using definition of DFT]

$$= \frac{1}{N} \sum_{m=0}^{N-1} x_1(m)\, x_2((k-m))_N$$

$$DFT\{x_1(n)\, x_2(n)\} = \frac{1}{N} [X_1(k) \circledast X_2(k)]$$

[∵ using definition of $\odot^{lar}$ conv.]

Circular Convolution:-

The $\odot^{lar}$ conv. of two N-point seq. $x_1(n)$ and $x_2(n)$ is defined as,

$$x_1(n) \circledast x_2(n) = \sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N$$

The Convolution property of DFT says that, the DFT of $\odot^{lar}$ conv. of 2 seq. is equivalent to product of their individual DFTs.

Let, $DFT\{x_1(n)\} = X_1(k)$ and $DFT\{x_2(n)\} = X_2(k)$ then by Convolution property, $DFT\{x_1(n) \circledast x_2(n)\} = X_1(k)\, X_2(k)$

Proof:-

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n) e^{\frac{-j2\pi nk}{N}} = \sum_{m=0}^{N-1} x_1(m) e^{\frac{-j2\pi mk}{N}}; \quad k = 0, 1, 2, \dots N-1 \quad \text{--- (1)}$$

$$X_2(k) = \sum_{n=0}^{N-1} x_2(n) e^{\frac{-j2\pi nk}{N}} = \sum_{p=0}^{N-1} x_2(p) e^{\frac{-j2\pi pk}{N}}; \quad k = 0, 1, 2, \dots N-1 \quad \text{--- (2)}$$

[let, n = m
let, n = p]

Consider the product $X_1(k)X_2(k)$. The IDFT of the product is given by.

$$DFT^{-1}\{X_1(k)X_2(k)\} = \frac{1}{N}\sum_{k=0}^{N-1} X_1(k)X_2(k)\,e^{\frac{j2\pi nk}{N}}$$

Sub. eqn ① & ② in above eqn. $X_1(k)$ & $X_2(k)$ We get,

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{m=0}^{N-1} x_1(m)\,e^{\frac{-j2\pi mk}{N}}\right]\left[\sum_{p=0}^{N-1} x_2(p)\,e^{\frac{-j2\pi pk}{N}}\right]e^{\frac{j2\pi nk}{N}}$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} x_1(m)\sum_{p=0}^{N-1} x_2(p)\sum_{k=0}^{N-1} e^{\frac{j2\pi k(n-m-p)}{N}} \quad \text{--- ③}$$

Consider the Summation $\sum_{k=0}^{N-1} e^{\frac{j2\pi k(n-m-p)}{N}}$ in eqn. ③

Let. $n-m-p = qN$. where q is an integer

Since q is an integer, $e^{j2\pi q} = 1$

$$\therefore \sum_{k=0}^{N-1} e^{\frac{j2\pi k(n-m-p)}{N}} = \sum_{k=0}^{N-1} e^{\frac{j2\pi kq\cancel{N}}{\cancel{N}}} = \sum_{k=0}^{N-1}\left(e^{j2\pi q}\right)^k$$

$$= \sum_{k=0}^{N-1} 1^k = N \quad \text{--- ④}$$

Consider the Summation $\sum_{p=0}^{N-1} x_2(p)$ in eqn. ③

Since, $n-m-p = qN$, $p = n-m-qN$

$$\therefore \sum_{p=0}^{N-1} x_2(p) = \sum_{m=0}^{N-1} x_2(n-m-qN) = \sum_{m=0}^{N-1} x_2(n-m,\ \text{mod } N)$$

$$= \sum_{m=0}^{N-1} x_2((n-m))_N \quad \text{--- ⑤}$$

Using eqn. ④ and ⑤, the eqn. ③ can be written as.

$$= \frac{1}{\cancel{N}}\sum_{m=0}^{N-1} x_1(m)\sum_{m=0}^{N-1} x_2((n-m))_N\,\cancel{N}$$

$$= \sum_{m=0}^{N-1} x_1(m)\,x_2((n-m))_N$$

$$DFT^{-1}\{X_1(k)X_2(k)\} = x_1(n) \circledast x_2(n)$$

$$\therefore X_1(k)X_2(k) = DFT\{x_1(n) \circledast x_2(n)\}$$

Parseval's Theorem:-

Let $DFT\{x_1(n)\} = X_1(k)$ and $DFT\{x_2(n)\} = X_2(k)$

Then by Parseval's relation,

$$\sum_{n=0}^{N-1} x_1(n)\,x_2^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_1(k)X_2^*(k)$$

Proof:-

Let $x_1(n)$ and $x_2(n)$ be N- Point Sequences.

Now by defn. of DFT $X_1(k) = \sum_{n=0}^{N-1} x_1(n)\,e^{\frac{-j2\pi nk}{N}}$ --- ①

Now by defn. of IDFT, $x_2(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_2(k)\,e^{\frac{j2\pi nk}{N}}$ --- ②

Consider the right - hand side term of Parseval's relation

$$\frac{1}{N}\sum_{k=0}^{N-1} X_1(k)X_2^*(k) = \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{n=0}^{N-1} x_1(n)\,e^{\frac{-j2\pi nk}{N}}\right]X_2^*(k)$$

By using Eqn. ①.

$$= \sum_{n=0}^{N-1} x_1(n)\left[\frac{1}{N}\sum_{k=0}^{N-1} X_2^*(k)\,e^{\frac{-j2\pi nk}{N}}\right]$$

$$= \sum_{n=0}^{N-1} x_1(n) \left[\frac{1}{N} \sum_{k=0}^{N-1} X_2(k) e^{\frac{j2\pi nk}{N}} \right]^* \quad \left[\because \text{From Eqn. (2)} \right]$$

$$\frac{1}{N} \sum_{k=0}^{N-1} X_1(k) X_2^*(k) = \sum_{n=0}^{N-1} x_1(n) x_2^*(n)$$

Hence Proved.

Fast Fourier Transform (FFT):-

It is a method (or algorithm) for computing the DFT with reduced no. of calculations. The computational efficiency is achieved if we adopt a divide and conquer approach.

Radix - 2 FFT:-

For radix-2 FFT, the value of N should be such that, $N = 2^m$, so that the N-point seq. is decimated into 2-point sequences and the 2-point DFT for each decimated sequence is computed. From the results of 2-pt. DFT, the 4-pt. DFTs can be computed. From the results of 4-pt. DFTs, the 8-pt. DFTs can be computed and so on, until we get N-point DFT.

Number of Calculations in Radix - 2 FFT:-

In radix-2 FFT, $N = 2^m$, and so there will be m stages of computations, where $m = \log_2 N$, with each stage having N/2 butterflies.

The no. of calculations in one butterfly are,

- No. of complex multiplication.
- No. of complex additions.

There are N/2 butterflies in each stage.

Therefore, no. of calculations in one stage are,

$N/2 \times 1 = N/2$ complex multiplications.

$N/2 \times 2 = N$ complex additions.

The N-point DFT involves m stages of computations. Therefore the no. of calculations for m stages are,

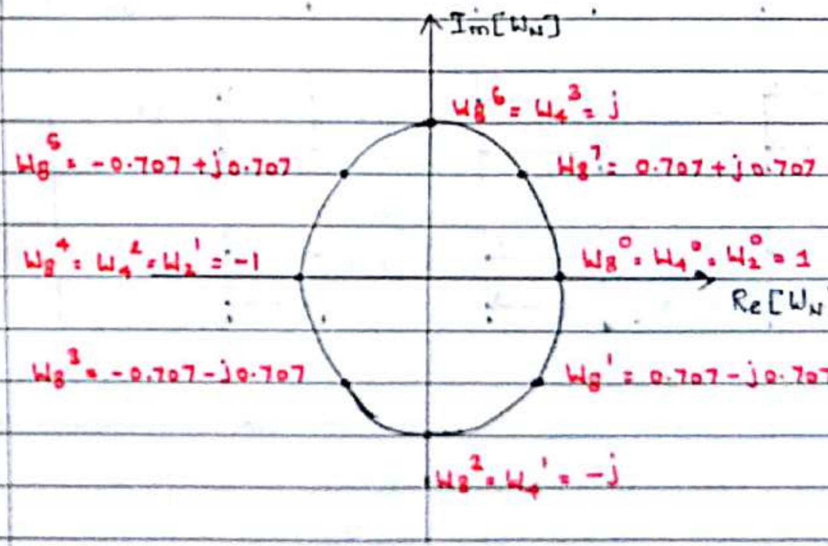
$$m \times \frac{N}{2} = \log_2 N \times \frac{N}{2} = \frac{N}{2} \log_2 N \text{ complex multiplications.}$$

$$m \times N = \log_2 N \times N = N \log_2 N \text{ complex additions.}$$

Phase (or) Twiddle factor:-

To simplify the notation it is desirable to define the complex valued phase factor W_N (also called as twiddle factor) which is an N^{th} root of unity as.

$$W_N = e^{\frac{-j2\pi}{N}}$$



$W_8^0 = W_4^0 = W_2^0 = 1$	$W_8^4 = W_4^2 = W_2^1 = -1$
$W_8^1 = 0.707 - j0.707$	$W_8^5 = -0.707 + j0.707$
$W_8^2 = W_4^1 = -j$	$W_8^6 = W_4^3 = j$
$W_8^3 = -0.707 - j0.707$	$W_8^7 = 0.707 + j0.707$

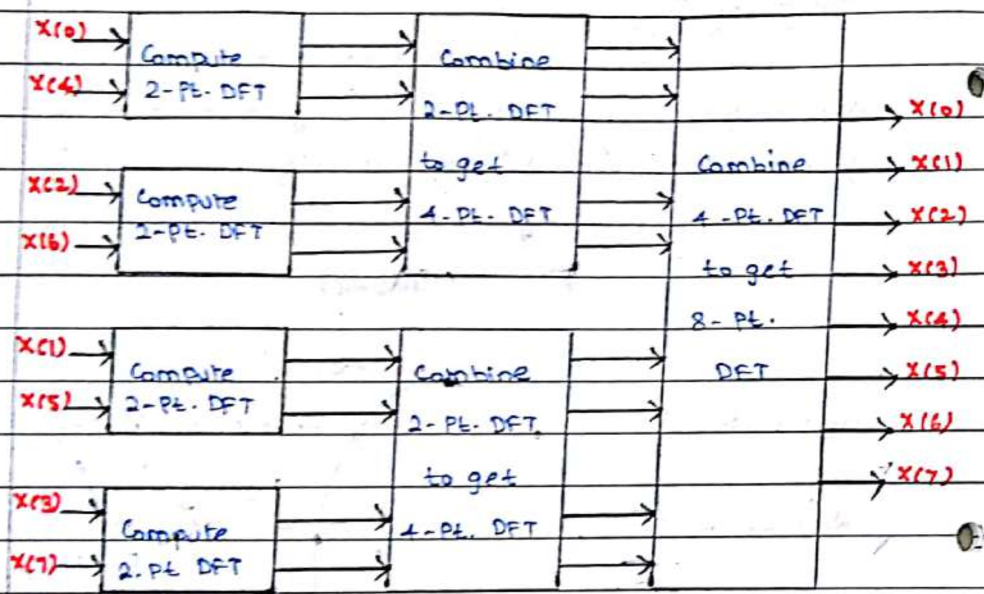
Decimation-in-Time Algorithm:- (DIT)

The I/p Seq. is 8-Pt. Sequence. Therefore, $N = 8 = 2^3 = r^m$.

Here, $r = 2$ and $m = 3$.

In DIT algorithm,

- ✓ The I/p is Bit reversed order.
- ✓ The o/p is Natural order.



I/p Sample Index	Natural order	Bit Reversed order	Bit Reversed Sample Index
0	000	000	0
1	001	100	4
2	010	010	2
3	011	110	6
⋮		⋮	⋮
7	111	111	7

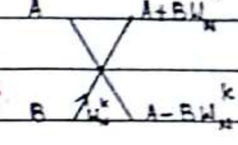
Problem:-

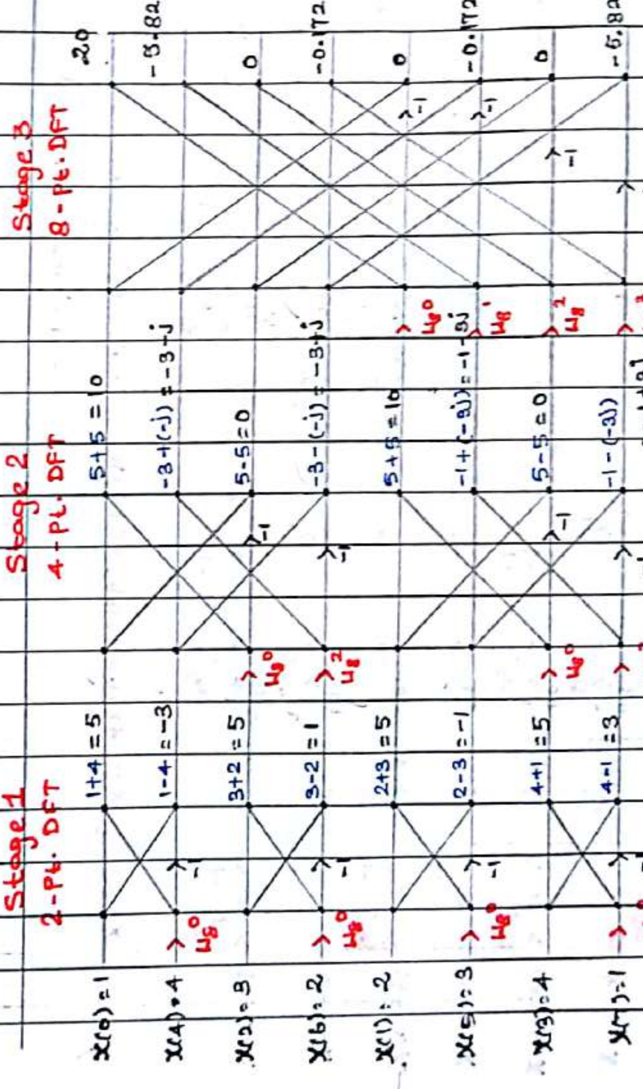
Find the DFT of the following Sequence using DIT Algorithm.

(1) $x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$

(PRB)

$x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$





$X(k) = \{20, -5.828 - j2.414, 0, -0.172 - j0.414, 0, -0.172 + j0.414, 0, -5.828 + j2.414\}$

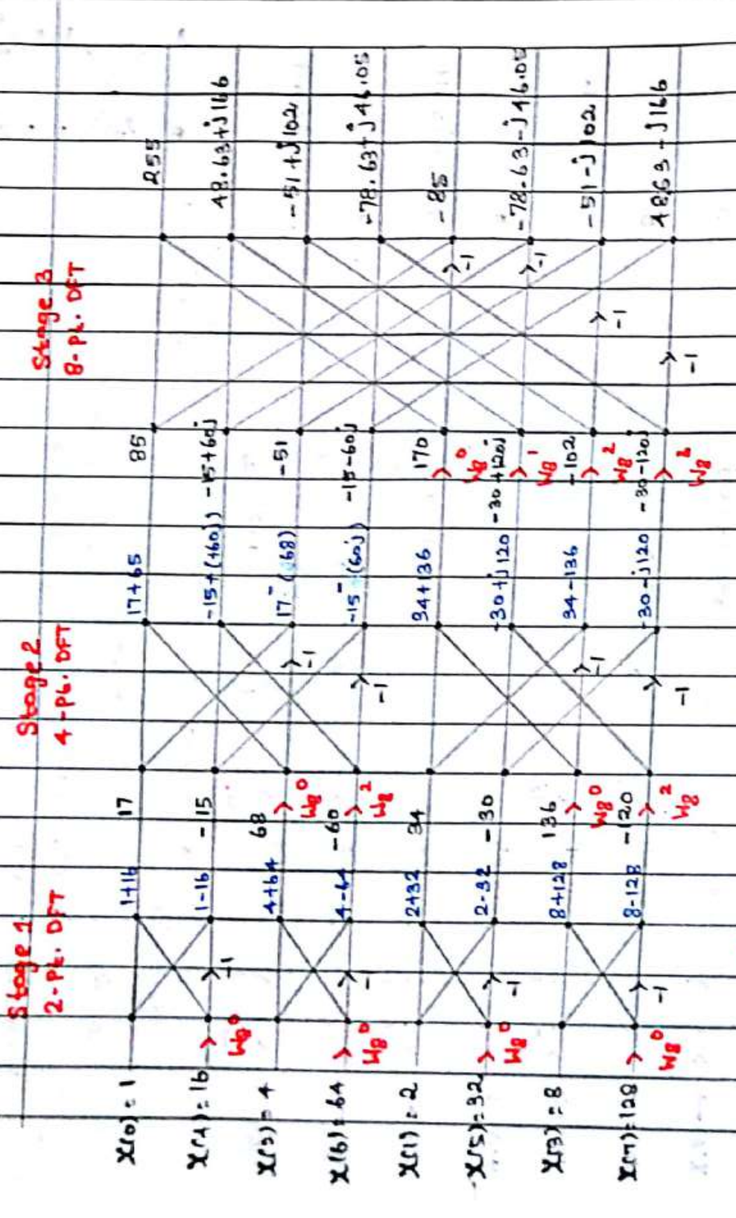
Stage 3 o/p:-

- ✓ $10 + 10 = 20$
- ✓ $(-3-j) + (0.707 - j0.707)(-1-3j) = -5.828 - j2.414$
- ✓ 0
- ✓ $(-3+j) + (-0.707 - j0.707)(-1+3j) = -0.172 - j0.414$
- ✓ 0
- ✓ $(-3-j) - (0.707 - j0.707)(-1-3j) = -0.172 + j0.414$
- ✓ 0
- ✓ $(-3+j) - (-0.707 - j0.707)(-1+3j) = -5.828 + j2.414$

(PRB)

(2) $x(n) = 2^n$ and $N=8$, find $X(k)$ using DIT

$x(n) = 2^n \quad \{1, 2, 4, 8, 16, 32, 64, 128\}$



$X(K) = \{255,\ 48.63 + j166,\ -51 + j102,\ -78.63 + j46.05,\ -85,\ -78.63 - j46.05,\ -51 - j102,\ 48.63 - j166\}$

Tutorial

(PRB)

(3) 8-Point DFT of the Sequence $x(n) = \{1,1,1,1,1,0,0,0\}$ using DIT-FFT algorithm.

$X(K) = \{5, -j2.414, 1, -0.414j, 1, 0.414j, 1, j2.414\}$

(PRB)

(4) Compute the 8-Point DFT of the Sequence using DIT-FFT algorithm.

$$x(n) = \begin{cases} 1 & 0 \le n \le 7 \\ 0 & \text{otherwise} \end{cases}$$

$X(K) = \{8, 0, 0, 0, 0, 0, 0, 0\}$

(PRB)

(5) Compute the 8-Point DFT of the Sequence using DIT-FFT algorithm.

$x(n) = \{0.5, 0.5, 0.5, 0.5, 0, 0, 0, 0\}$

$X(K) = \{2, 0.5 - j1.207, 0, 0.5 - j0.207, 0, 0.5 + j0.207, 0, 0.5 + j1.207$

(ANK)

(6) $x(n) = \{2, 1, 2, 1, 1, 2, 1, 2\}$ using DIT-FFT algorithm.

$X(K) = \{12, 1 + j0.414, 0, 1 + j2.414, 0, 1 - j2.414, 0, 1 - j0.414\}$

Decimation in Frequency Algorithm :- (DIF)

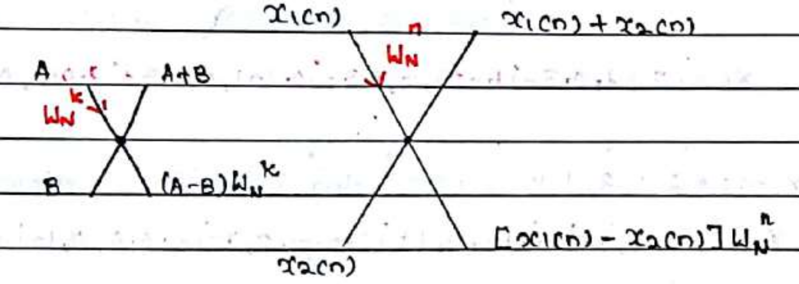
In DIF algorithm the frequency domain seq. $X(k)$ is decimated. In this algorithm, the N-pt. time domain seq. is converted to two numbers of N/2 point sequences.

Then each N/2 pt. seq. is converted to 2 numbers of N/4 pt. sequences. In this algorithm, the I/p seq $x(n)$ is partitioned into 2 seq. each of length N/2 samples.

The first seq $x_1(n)$ consists of first N/2 samples of $x(n)$ and the second seq. $x_2(n)$ consists of the last N/2 samples of $x(n)$ i.e.,

$x_1(n) = x(n), \; n = 0, 1, 2 \ldots N/2 - 1$

$x_2(n) = x(n + N/2), \; n = 0, 1, 2 \ldots N/2 - 1$



I/p Sample Index	Natural order	Bit reverse order	Bit Reverse Sample Index
0	000	000	0
1	001	100	4
2	010	010	2
⋮	⋮	⋮	⋮
7	111	111	7

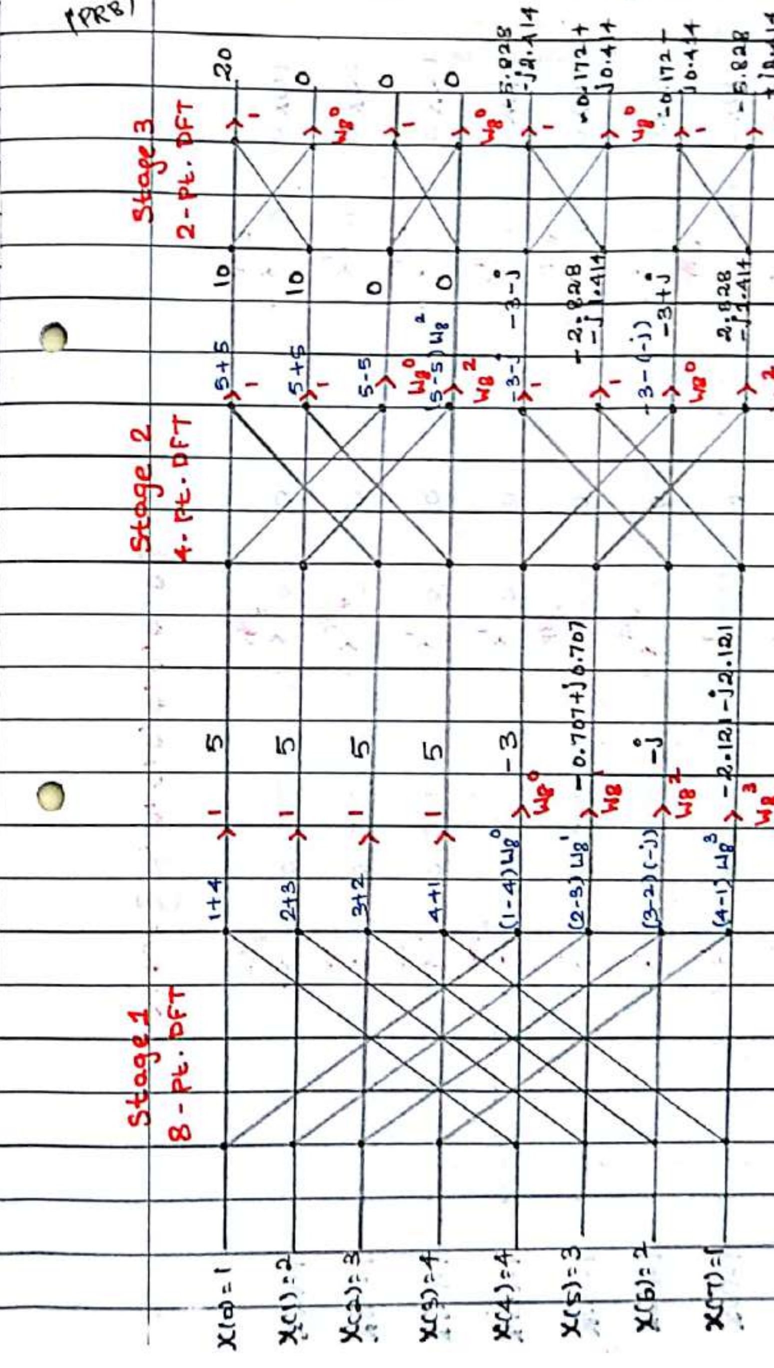
✓ I/p is Natural order

✓ O/p is Bit reversed order.

Problem:-

(1) $x(n): \{1, 2, 3, 4, 4, 3, 2, 1\}$ using DIF-FFT Algorithm.

(PR8)



Stage 1:-

$= (2-3)(0.707 - j0.707) = -0.707 + j0.707$

$= (4-1)(-0.707 - j0.707) = -2.121 - j2.121$

Stage 2:-

$= -0.707 + j0.707 + (-2.121 - j2.121) = -2.828 - j1.414$

$= (-0.707 + j0.707 - (-2.121 - j2.121))(-j)$

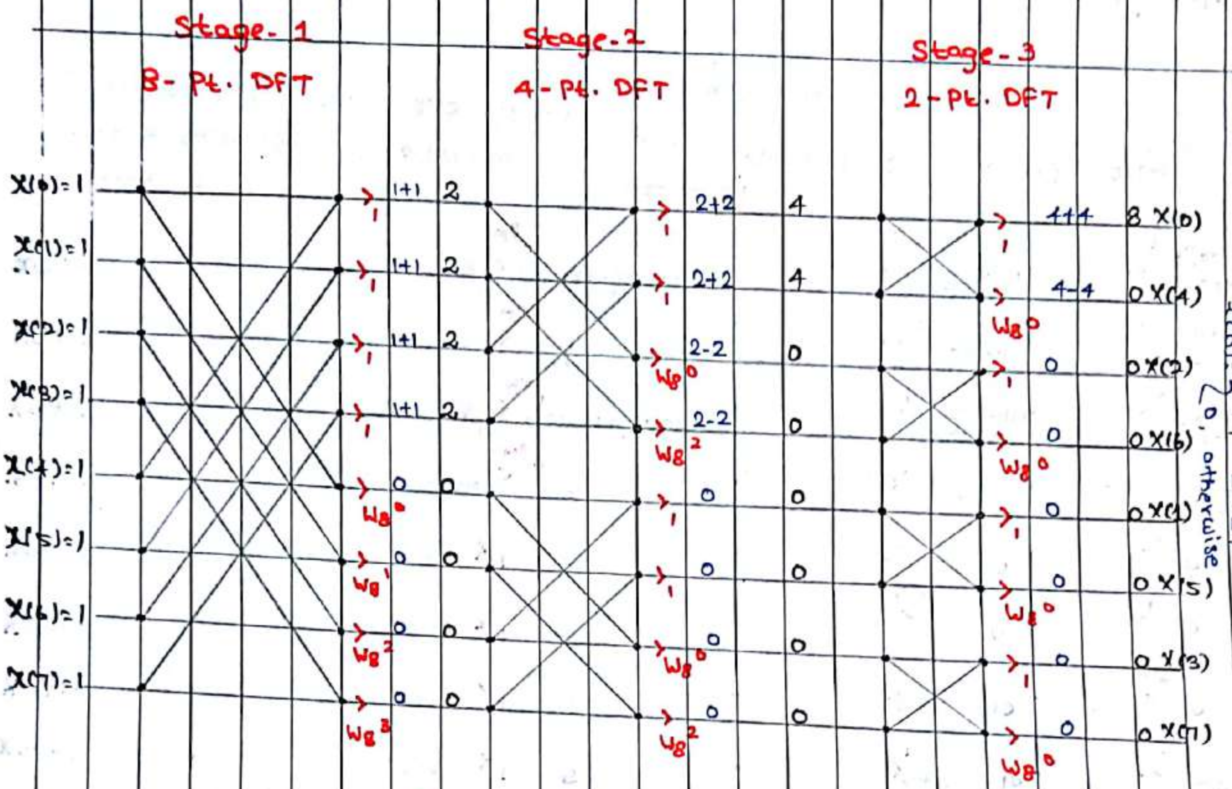
$= 2.828 - j1.414$

$X(k) = \{20, -5.828 - j2.414, 0, -0.172 - j0.414, 0, -0.172 + j0.414, 0, -5.828 + j2.414\}$

(O/p Bit Reversed order)

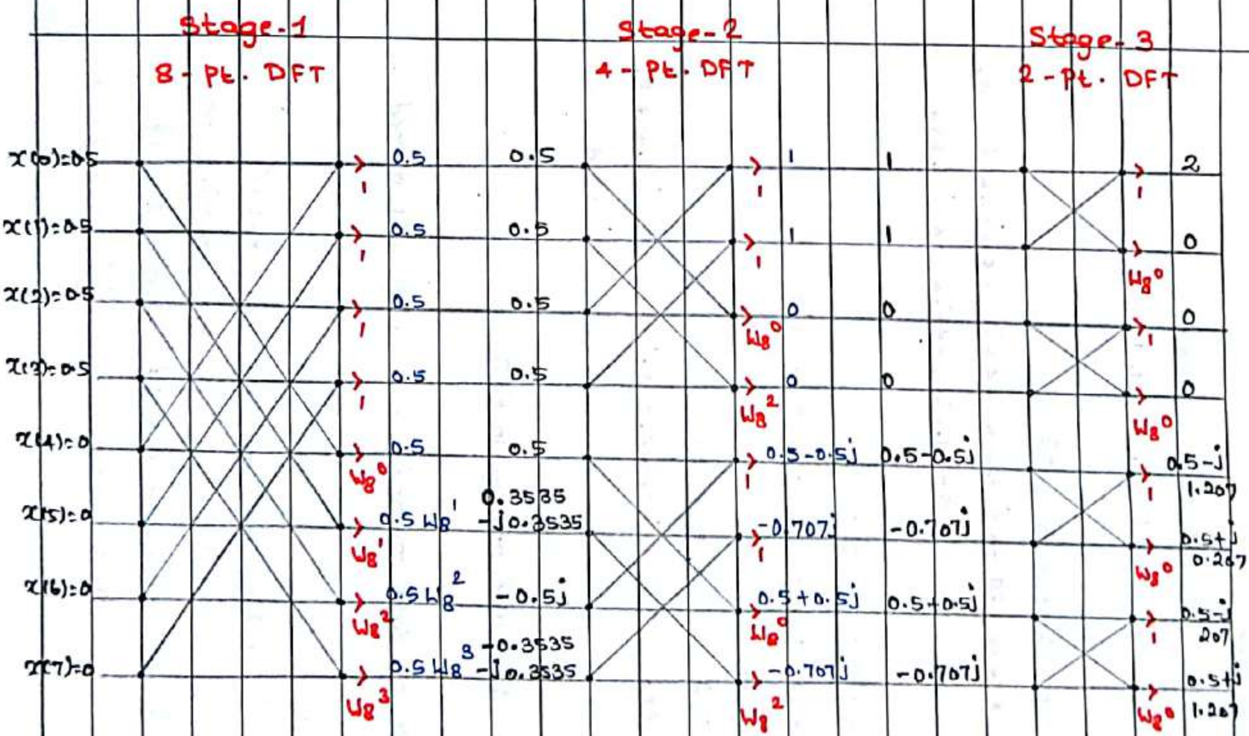
(PRB) (2) Compute 8-pt. DFT of the sequence by using DIF algorithm.

$x(n) = \begin{cases} 1, & 0 \le n \le 7 \\ 0, & \text{otherwise} \end{cases}$



$X(k) = \{8, 0, 0, 0, 0, 0, 0, 0\}$

(PRB) (3) Compute 8-pt. DFT $x(n) = \{0.5, 0.5, 0.5, 0.5, 0, 0, 0, 0\}$ using DIF algorithm



Stage 1:-

$= 0.5(0.707 - j0.707) = 0.3535 - j0.3535$

Stage 2:-

$= 0.5 + (-0.5j) = 0.5 - 0.5j$

$= (0.3535 - j0.3535) + (-0.3535 - j0.3535)$

$= -0.707j$

$= 0.5 - (-0.5j) = 0.5 + 0.5j$

$X(k) = \{2, 0.5 - j1.207, 0, 0.5 - j0.207, 0, 0.5 + j0.207, 0, 0.5 + j1.207\}$

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Tutorial (4) Compute the 8-Pt. DFT of the following sequences using DIF algorithm.

(PRB)

$x(n) = \{1,0,0,0,0,0,0,0\}$

$X(k) = \{1,1,1,1,1,1,1,1\}$

(PRB) (5) $x(n) = \{0,1,2,3,4,5,6,7\}$

$X(k) = \{28, -4+j9.656, -4+j4, -4+j1.656, -4, -4-j1.656, -4-j4, -4-j9.656\}$

(PRB) (6) $x(n) = \{1,2,2,1,1,2,2,1\}$

$X(k) = \{12, 0, -2-j2, 0, 0, 0, -2+j2, 0\}$

(ANK) (7) $x(n) = \{2,1,2,1,1,2,1,2\}$

$X(k) = \{12, 1+j0.414, 0, 1+j2.414, 0, 1-j2.414, 0, 1-j0.414\}$

Use of Linear filtering in FFT:-

✓ overlap save method ✓ overlap add method

Tutorials

Problems:-

(ANK) (1) $x(n) = \{1,-1,2,-2,3,-3,4,-4\}$; $h(n) = \{-1,1\}$

$y(n) = \{-1,2,-3,4,-5,6,-7,8,-4\}$

(ANK) (2) $x(n) = \{1,2,3,-1,-2,-3,4,5,6\}$; $h(n) = \{2,1,-1\}$

$y(n) = \{2,5,7,-1,-8,-7,7,17,13,1,-6\}$

(P.T.O)

Discrete Cosine Transform: (DCT)

A DCT expresses a finite sequence of data points in terms of a sum of cosine functions oscillating at different frequencies. In particular, a DCT is a fourier related transform similar to the DFT, but using only real numbers.

It is widely used for data compression. Similar to FFT, DCT converts data into set of frequencies. It operates on a function at a finite number of discrete data points.

$$X(k) = e(k) \sum_{n=0}^{N-1} x(n) \cos\left[\frac{(2n+1)k\pi}{2N}\right] \quad k = 0,1,2,\dots N-1$$

Where,

$$e(k) = \begin{cases} 1/\sqrt{2}, & \text{if } k=0 \\ 1, & \text{otherwise} \end{cases}$$

Homework

To be continued...

(PRB) (3) $x(n) = \{1,2,-1,2,3,-2,-3,-1,1,1,2,-1\}$; $h(n) = \{1,2\}$

$y(n) = \{1,4,3,0,7,4,-7,-7,-1,3,4,3,-2\}$

(PRB) (4) $x(n) = \{1,-1,1,2,1,0,1,-4,3,2,1,0,1,1\}$; $h(n) = \{1,1,2,1\}$

$y(n) = \{1,0,2,2,4,6,5,-2,1,-2,5,8,5,3,3,3,1\}$

Convolution

- Linear convolution
  - Matrix method
  - Tabular method
  - Graphical method
- Circular convolution
  - Matrix method
  - Concentric circle method

Linear convolution using matrix method :-

Let $x(n) = \{1,2,3,4\}$
$h(n) = \{1,2,1,2\}$

$x(n), h(n)$ length must not be same in linear convolution and $y(n)$ length should be satisfied by the eqn $N = L+M-1$

$y(n) = x(n) * h(n)$

x(n) \ h(n)	1	2	1	2
1	1	2	1	2
2	2	4	2	4
3	3	6	3	6
4	4	8	4	8

$\therefore y(n) = \{1,4,8,14,15,10,8\}$

$N = L+M-1$

where N is length of o/p signal
L is length of i/p signal
M is length of impulse

$\therefore N = 4+4-1 = 7$

Linear convolution using tabular method:-

$$
\begin{array}{r}
1\ 2\ 3\ 4 \\
1\ 2\ 1\ 2 \\
\hline
2\ 4\ 6\ 8 \\
1\ 2\ 3\ 4\ \ \\
2\ 4\ 6\ 8\ \ \ \ \\
1\ 2\ 3\ 4\ \ \ \ \ \ \\
\hline
1\ 4\ 8\ 14\ 15\ 10\ 8 \\
\hline
\end{array}
$$

$\therefore y(n) = \{1, 4, 8, 14, 15, 10, 8\}$

Circular convolution method: The sequence, length of $x(n)$ and $h(n)$ must be same. If $x(n)$ and $h(n)$ sequence is different, you can pad zeroes

Mostly the length of $y(n)$ will be equal to length of $x(n)$ or $h(n)$

Circular convolution using matrix method:

Let $x(n) = \{1, 2, 3, 4\}$
$h(n) = \{5, 6, 7, 8\}$

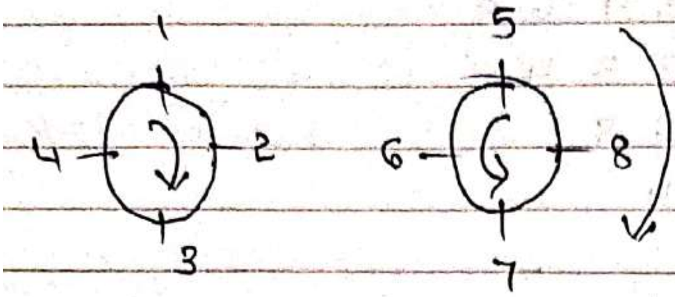
$$
\begin{array}{cccc}
h(n) & x(n) & & & y(n) \\
\begin{bmatrix} 5 & 8 & 7 & 6 \\ 6 & 5 & 8 & 7 \\ 7 & 6 & 5 & 8 \\ 8 & 7 & 6 & 5 \end{bmatrix} &
\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} & = &
\begin{bmatrix} 5+16+21+24 \\ 6+10+24+28 \\ 7+12+15+32 \\ 8+14+18+20 \end{bmatrix} = &
\begin{bmatrix} 66 \\ 68 \\ 66 \\ 60 \end{bmatrix}
\end{array}
$$

$\therefore y(n) = \{66, 68, 66, 60\}$

The length of $y(n)$ is 4.

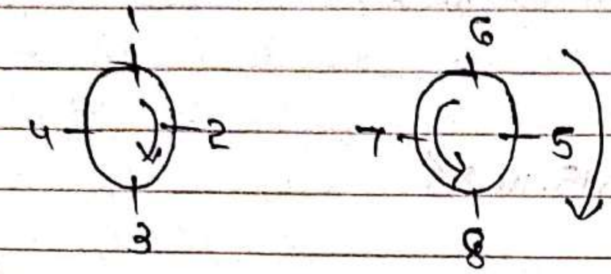
Concentric circular method :-

x(n)   h(n)   y(n)



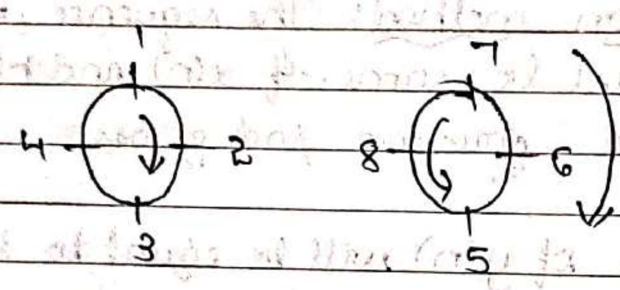
$= 5+16+21+24$

$= 66$



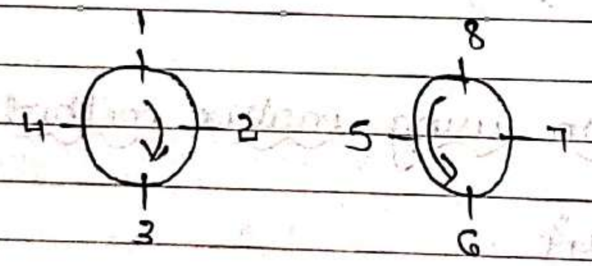
$= 6+10+24+28$

$= 68$



$= 7+12+15+32$

$= 66$



$= 8+14+18+20$

$= 60$

$\therefore y(n) = \{66, 68, 66, 60\}$

The length of y(n) is 4

Linear convolution using circular matrix method:

Let $x(n) = \{1,2,3,4\}$

$h(n) = \{5,6,7,8\}$

W.K.t $N = L+M-1$

$= 4+4-1$

$\therefore N = 7$

$\therefore x(n) = \{1,2,3,4,0,0,0\}$

$h(n) = \{5,6,7,8,0,0,0\}$

$$\underbrace{\begin{bmatrix} 5 & 0 & 0 & 0 & 8 & 7 & 6 \\ 6 & 5 & 0 & 0 & 0 & 8 & 7 \\ 7 & 6 & 5 & 0 & 0 & 0 & 8 \\ 8 & 7 & 6 & 5 & 0 & 0 & 0 \\ 0 & 8 & 7 & 6 & 5 & 0 & 0 \\ 0 & 0 & 8 & 7 & 6 & 5 & 0 \\ 0 & 0 & 0 & 8 & 7 & 6 & 5 \end{bmatrix}}_{h(n)} \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{x(n)} = \underbrace{\begin{bmatrix} 5+0 \\ 6+10 \\ 7+12+15 \\ 8+14+18+20 \\ 16+21+24 \\ 24+28 \\ 32 \end{bmatrix}}_{y(n)} = \underbrace{\begin{bmatrix} 5 \\ 16 \\ 34 \\ 60 \\ 61 \\ 52 \\ 32 \end{bmatrix}}_{y(n)}$$

By linear convolution [matrix method],

x(n) \ h(n)	5	6	7	8
1	5	6	7	8
2	10	12	14	16
3	15	18	21	24
4	20	24	28	32

$\therefore y(n) = \{5, 16, 34, 60, 61, 52, 32\}$

Correlation:- It is a measure of the degree to which 2 signals are similar. The correlation of 2 signals is divided into auto correlation and cross correlation.

Cross correlation:- The cross correlation between a pair of signal $x(n)$ and $y(n)$ is given by

$$\gamma_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)y(n-l) \; ; \; l = 0, \pm 1, \pm 2, \ldots \rightarrow 1$$

The index l is the shift (lag) parameter

$$\gamma_{xy}(l \quad \gamma_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n)\,x(n-l) \; ; \; l = 0, \pm 1, \pm 2, \ldots$$

$$\gamma_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)\,y(n+l) \rightarrow 2$$

If time shift, $l = 0$

$$\gamma_{xy}(0) = \sum_{n=-\infty}^{\infty} x(n)\,y(n) \rightarrow 3$$

Comparing eqn 1, 2 and 3

$$r_{xy}(l) = r_{yx}(-l)$$

Auto-correlation:- The auto-correlation of a sequence is correlation of a sequence with itself. The auto-correlation of a sequence is given by $r_{xx}(l)$.

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)\,x(n-l) \;;\; l = 0, \pm 1, \pm 2, \ldots$$

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)\,x(n+l)$$

If $l = 0$, $r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n).x(n) = \sum_{n=-\infty}^{\infty} x^2(n)$

Properties of correlation:

* The cross-correlation sequence $r_{xy}(l)$ is simply a folded version of $r_{yx}(l)$ i.e $r_{xy}(l) = r_{yx}(-l)$.
* Similarly for auto correlation, $r_{xx}(l) = r_{xx}(-l)$
* Auto correlation is an even function.

The cross-correlation function satisfies the condition,

$$|r_{xy}(l)| \leq \sqrt{r_{xx}(0)\,r_{yy}(0)} = \sqrt{E_x E_y}$$

where E_x and E_y are energies of $x(n)$ and $y(n)$ respectively.

On applying the above condition to auto correlation we get,

$$|r_{xx}(l)| \leq \sqrt{r_{xx}(0)\,\cancel{r_{yy}}\,r_{xx}(0)}$$

$$\therefore |r_{xx}(l)| \leq r_{xx}(0) = E_x$$

* Using the max value of cross correlation sequence, the normalized cross correlation sequence is defined as

$$\rho_{xy}(l) = \frac{r_{xy}(l)}{\sqrt{r_{xx}(0)\, r_{yy}(0)}}$$

Similarly for auto correlation,

$$\rho_{xx}(l) = \frac{r_{xx}(l)}{r_{xx}(0)}$$

1. Find the auto correlation of sequence $x(n) = \{1,2,3,4,5\}$

Sol: Let $x(n) = \{\underset{\uparrow}{1},2,3,4,5\}$

$x(-n) = \{\underset{\uparrow}{5},4,3,2,1\}$

$x(n)$ \ $x(-n)$	5	4	3	2	1
1	5	4	3	2	1
2	10	8	6	4	2
3	15	12	9	6	3
4	20	16	12	8	4
5	25	20	15	10	5

$\therefore r_{xx}(n) = \{5, 14, 26, 40, 55, 40, 26, 14, 5\}$

2. Find the cross correlation sequence of the following given signal $x(n) = \{1,2,1,1\}$, $h(n) = \{1,2,2,1\}$

Sol: Given $x(n) = \{1,2,1,1\}$

$h(n) = \{1,2,2,1\}$

then $h(-n) = \{1,2,2,1\}$

*Cross correlation ans will be equal to linear convolution ans

$x(n) \backslash h(-n)$	1	2	2	1
1	1	2	2	1
2	2	4	4	2
1	1	2	2	1
1	1	2	2	1

$$r_{xh}(n) = \{1, 4, 7, 8, 8, 3, 1\}$$

Linear convolution,

$x(n) \backslash h(n)$	1	2	2	1
1	1	2	2	1
2	2	4	4	2
1	1	2	2	1
1	1	2	2	1

$$y(n) = \{1, 4, 7, 8, 8, 3, 1\}$$

3. $x(n) = \{3, 0, 3\}$, $h(n) = \{-3, 1, -3\}$, $r_{xh}(n) = ?$

Sol:

$x(n) \backslash h(-n)$	-3	1	-3
3	-9	3	-3
0	0	0	0
3	-9	3	-9

$$r_{xh}(n) = \{-9, 3, -12, 3, -9\}$$

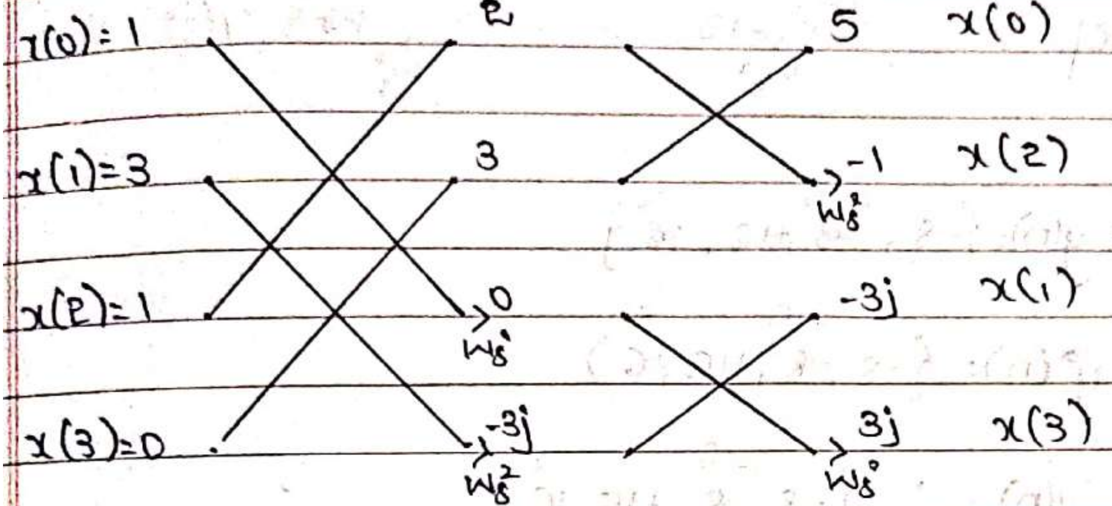
$x(n) \backslash h(n)$	-3	1	-3
3	-9	3	-9
0	0	0	0
3	-9	3	-9

$$y(n) = \{-9, 3, -12, 3, -9\}$$

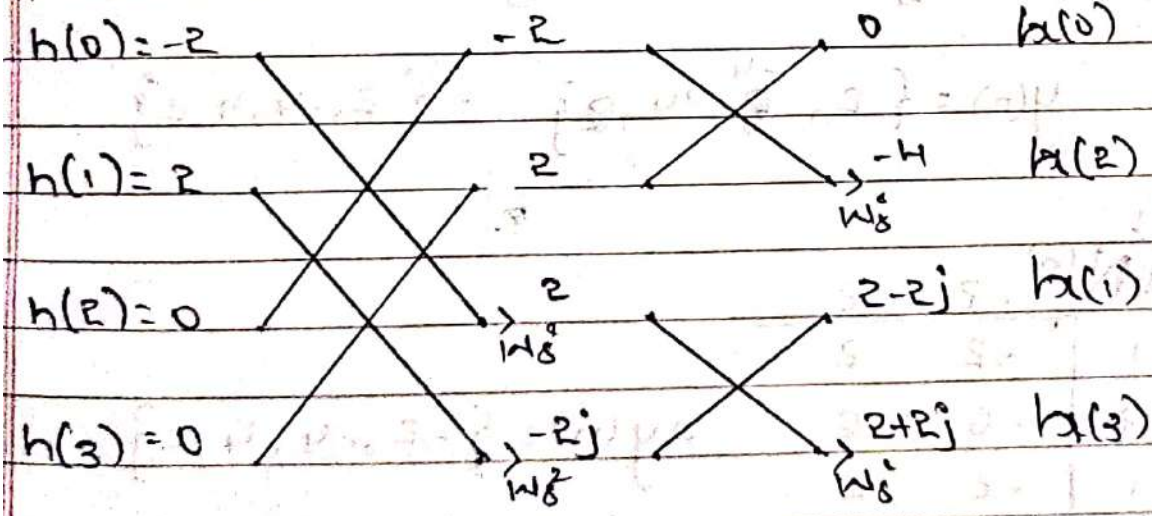
Use of linear filtering in FFT :-

Compute the linear convolution of the following sequence by using FFT method. $x(n) = \{1,3,1\}$ and $h(n) = \{-2,2\}$

$x(n) = \{1,3,1,0\}$ and $h(n) = \{-2,2,0,0\}$



$\therefore X(k) = \{5, -3j, -1, 3j\}$

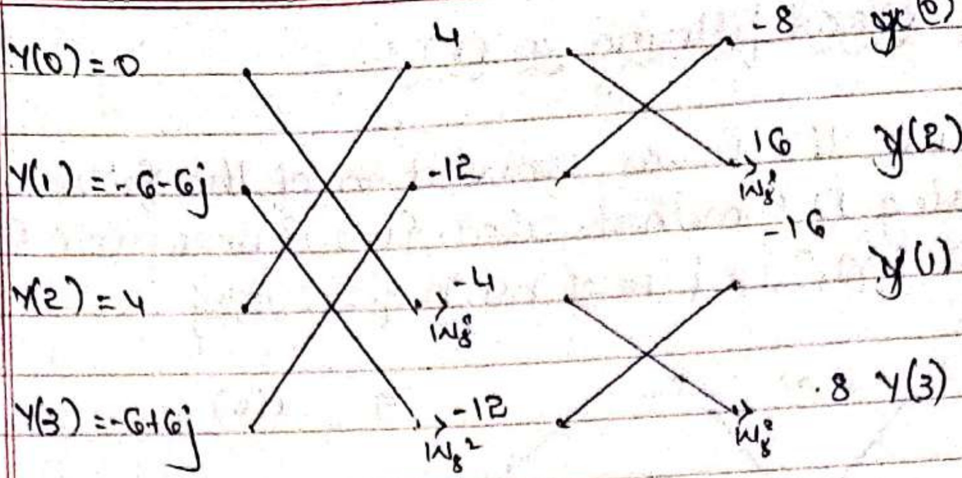


$\therefore H(k) = (0, 2-2j, -4, 2+2j)$

$Y(k) = X(k) * H(k)$

$= \{0, -6+6j, 4, -6-6j\}$

$Y^*(k) = \{0, -6-6j, 4, -6+6j\}$



$\therefore y(n) = \{-8, -16, +16, 8\}$

$y^*(n) = \{-8, -16, +16, 8\}$

$y(n) = \frac{1}{N}\{-8, -16, +16, 8\}$

$y(n) = \frac{1}{4}\{-8, -16, +16, 8\}$

$y(n) = \{2, -4, +4, 2\} \quad = \{-2, -4, 4, 2\}$

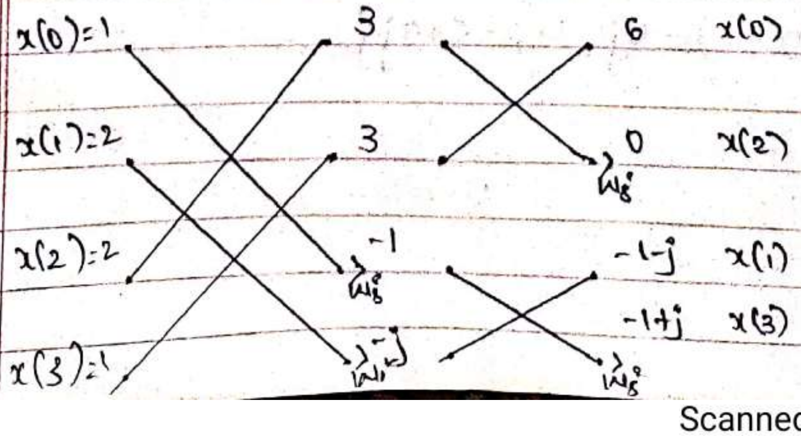
Proof:-

$x(n) \backslash h(n)$	-2	2
1	-2	2
3	-6	6
1	-2	2

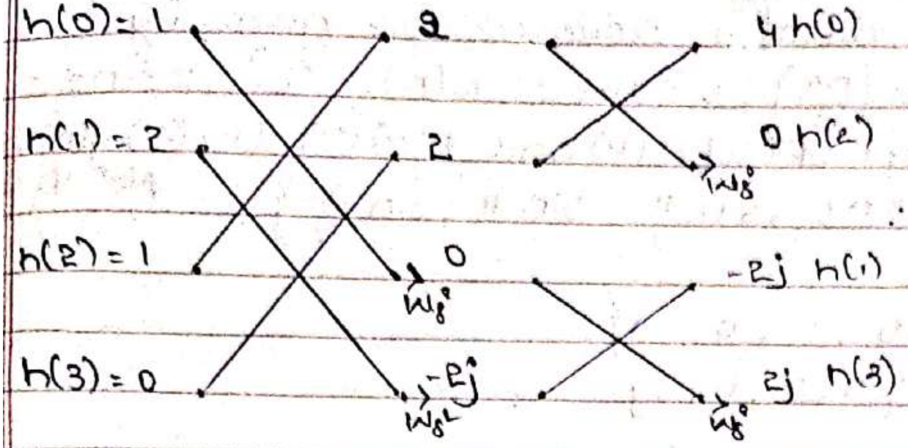
$\Rightarrow y(n) = \{-2, -4, 4, 2\}$

2. $x(n) = \{1,2,2,1\}, h(n) = \{1,2,1\}$

Sol: $x(n) = \{1,2,2,1\}, h(n) = \{1,2,1,0\}$



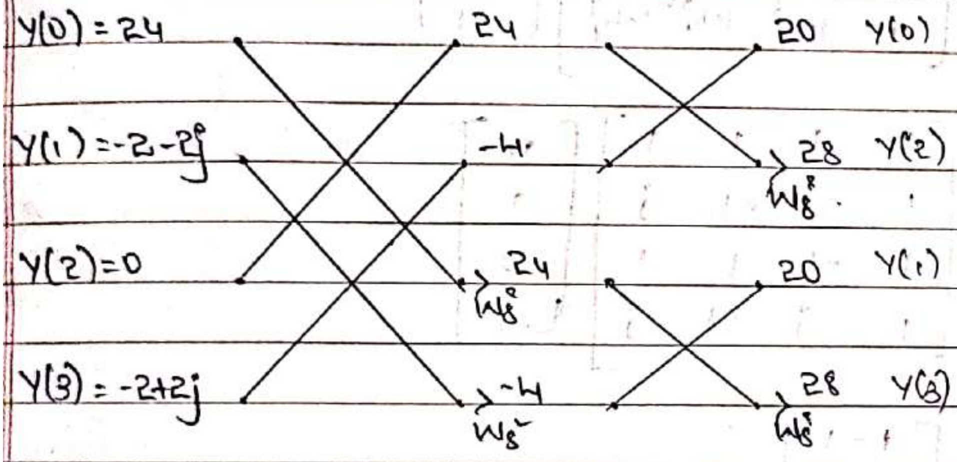
$\therefore X(k) = \{6, -1-j, 0, -1+j\}$



$\therefore H(k) = \{4, -2j, 0, 2j\}$

$Y(k) = X(k) * H(k) = \{24, -2j+2, 0, -2j-2\}$

$Y^*(k) = \{24, -2j-2, 0, -2+2j\}$



$= \{20, 20, 28, 28\}$

$y(n) = \frac{1}{4}\{20, 20, 28, 28\}$

$\therefore y(n) = \{5, 5, 7, 7\}$

x(n) \ h(n)	1	2	1
1	1	2	1
2	2	4	2
2	2	4	2
1	1	2	1

$\therefore \{1, 4, 7, 7, 4, 1\}$

$\therefore y(n) = \{5, 5, 7, 7\}$

Linear filtering using DFT (or) Circular convolution using DFT and IDFT:-

(1.) Two finite duration sequences are given by

$x(n) = \{\sin(\frac{n\pi}{2}); 0 \le n \le 3, h(n) = 2^n; 0 \le n \le 3$

Calculate (i) 4 pt DFT of X(k) and H(k) (ii) Y(k) (iii) y(n) and plot the graph

Sol: $x(n) = \{\sin 0, \sin\frac{\pi}{2}, \sin\pi, \sin\frac{3\pi}{2}\}$

$\therefore x(n) = \{0, 1, 0, -1\}$

$h(n) = \{1, 2, 4, 8\}$

$$X(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^{6} & W_4^9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-1 \\ -j-j \\ -1+1 \\ j+j \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -2j \\ 0 \\ 2j \end{bmatrix}$$

$\therefore X(k) = \{0, -2j, 0, 2j\}$

$$H(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 1+2+4+8 \\ 1-2j-4+8j \\ 1-2+4-8 \\ 1+2j-4-8j \end{bmatrix} = \begin{bmatrix} 15 \\ -3+6j \\ -5 \\ -3-6j \end{bmatrix}$$

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$\therefore H(k) = \{15, -3+6j, -5, -3-6j\}$

$Y(k) = X(k) \cdot H(k)$

$= \{0, 6j+12, 0, -6j+12\}$

$\therefore Y(k) = \{0, 12+6j, 0, 12-6j\}$

$Y^*(k) = \{0, 12-6j, 0, 12+6j\}$

$$y(n) = \frac{1}{N}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}\begin{bmatrix} 0 \\ 12-6j \\ 0 \\ 12+6j \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 12-\cancel{6j}+12+\cancel{6j} \\ -\cancel{12j}-6+\cancel{12j}-6 \\ -12+\cancel{6j}-12-\cancel{6j} \\ \cancel{12j}+6-\cancel{12j}+6 \end{bmatrix}^*$$

$$= \frac{1}{4}\begin{bmatrix} 24 \\ -12 \\ -24 \\ 12 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -3 \\ -6 \\ 3 \end{bmatrix}$$

$\therefore y(n) = \{6, -3, -6, 3\}$

Filtering of long data sequences:-

Overlap add and Overlap save method:- An important application of FFT is FIR linear filtering of long data sequence. There are 2 methods based on the uses of FFT algorithm.

1) Overlap save method
2) Overlap add method

Overlap save method :- Let the length of the i/p sequence $x(n)$ is L and the length of the impulse response $h(n)$ is M.

* i/p is divided into blocks of data of size $N = L + M - 1$

* Each block consists of last data points $(M-1)$ of previous block followed by L new data points to form a data sequence of length $N = L + M - 1$

* For first block of data the first $(M-1)$ points are said to be zero.

* Now the impulse response of FIR filter is increased in the length by appending or padding $(L-1)$ zeros and an N point circular convolution of $x(n)$ and $h(n)$ is computed.

$$y(n) = x(n) \circledN h(n)$$

$$x_1(n) = \{\underbrace{0, 0, 0, \ldots, 0}_{(M-1)\text{ zeros}}, \underbrace{x(0), x(1), \ldots, x(L-1)}_{\text{'L' new data points}}\}$$

$$x_2(n) = \{\underbrace{x(L-M+1), \ldots, x(L-1)}_{\substack{(M-1)\text{ data points from } x_1(n) \\ [\text{Last } (M-1) \text{ datas}]}}, \underbrace{x(L), \ldots, x(2L-1)}_{\text{'L' new data points}}\}$$

$$x_3(n) = \{\underbrace{x(2L-M+1), \ldots, x(2L-1)}_{\substack{(M-1)\text{ data points} \\ \text{from } x_2(n)}}, \underbrace{x(2L), x(2L+1), \ldots, x(3L-1)}_{\substack{\text{'L' new data points and} \\ \text{so on.}}}\}$$

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1. Find the o/p $y(n)$ of the filter whose impulse response $h(n)=\{1,1,1\}$ and i/p signal $x(n)=\{3,-1,0,1,3,2,0,1,2,1\}$ using overlap save method.

Sol: $N=L+M-1 \Rightarrow N=10+3-1=12$

For reference, $N=5$

$\Rightarrow 5 = L+3-1$

$\Rightarrow L=3$

$x_1(n)=\{0,0,3,-1,0\}$

$x_2(n)=\{-1,0,1,3,2\}$

$x_3(n)=\{3,2,0,1,2\}$

$x_4(n)=\{1,2,1,0,0\}$

$h(n)=\{1,1,1,0,0\}$

$$y_1(n)=\underbrace{\begin{bmatrix}1&0&0&1&1\\1&1&0&0&1\\1&1&1&0&0\\0&1&1&1&0\\0&0&1&1&1\end{bmatrix}}_{h(n)}\underbrace{\begin{bmatrix}0\\0\\3\\-1\\0\end{bmatrix}}_{x_1(n)}=\begin{bmatrix}-1\\0\\3\\2\\2\end{bmatrix}$$

$$y_2(n)=\underbrace{\begin{bmatrix}1&0&0&1&1\\1&1&0&0&1\\1&1&1&0&0\\0&1&1&1&0\\0&0&1&1&1\end{bmatrix}}_{h(n)}\begin{matrix}\cancel{0}\\\cancel{0}\\\cancel{3}\\\cancel{-1}\\\cancel{0}\end{matrix}\underbrace{\begin{bmatrix}-1\\0\\1\\3\\2\end{bmatrix}}_{x_2(n)}=\begin{bmatrix}4\\1\\0\\4\\6\end{bmatrix}$$

$$y_3(n)=\underbrace{\begin{bmatrix}1&0&0&1&1\\1&1&0&0&1\\1&1&1&0&0\\0&1&1&1&0\\0&0&1&1&1\end{bmatrix}}_{h(n)}\underbrace{\begin{bmatrix}3\\2\\0\\1\\2\end{bmatrix}}_{x_3(n)}=\begin{bmatrix}6\\7\\5\\3\\3\end{bmatrix}$$

$x_4(n)$ \ $h(n)$	1	1	1
1	1	1	1
2	2	2	2
1	1	1	1
0	0	0	0
0	0	0	0

$= \{1, 3, 4, 3, 1, 0, 0\}$

$\therefore y_4(n) = \{1, 3, 4, 3, 1\}$

$\therefore y(n) = \{3, 2, 2, 0, 4, 6, 5, 3, 3, 4, 3, 1\}$

2. $x(n) = \{1, -1, 2, -2, 3, -3, 4, -4\}$, $h(n) = \{-1, 1\}$

Sol: $N = L + M - 1 = 8 + 2 - 1 = 9$

For reference, $N = 5$

$5 = L + 2 - 1$

$\therefore L = 4$

$x_1(n) = \{0, 1, -1, 2, -2\}$ $\quad x_3(n) = \{-4, 0, 0, 0, 0\}$

$x_2(n) = \{-2, 3, -3, 4, -4\}$

$h(n) = \{-1, 1, 0, 0, 0\}$

$$y_1(n) = \overset{h(n)}{\begin{bmatrix} -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}} \overset{x_1(n)}{\begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \\ -2 \end{bmatrix}} = \begin{bmatrix} -2 \\ -1 \\ 2 \\ -3 \\ 4 \end{bmatrix}$$

$\therefore y_1(n) = \{-2, -1, 2, -3, 4\}$

$$y_2(n) = \overset{h(n)}{\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}} \overset{x_2(n)}{\begin{bmatrix} -2 \\ 3 \\ -3 \\ 4 \\ -4 \end{bmatrix}} = \begin{bmatrix} -6 \\ -5 \\ 6 \\ -7 \\ 8 \end{bmatrix}$$

$\therefore y_2(n) = \{-6, -5, 6, 7, 8\}$

$$y_3(n) = \begin{array}{c} h(n) \\ \begin{bmatrix} -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \end{array} \begin{array}{c} x_3(n) \\ \begin{bmatrix} -4 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array} = \begin{bmatrix} 4 \\ -4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore y_3(n) = \{\underset{\uparrow}{4}, -4, 0, 0, 0\}$$

$$\therefore y(n) = \{-1, 2, -3, 4, -5, 6, -7, 8, -4\}$$

Overlap add method:-

i/p sequence length = L

Impulse response length = M

No. of zeros padding = M-1

The o/p sequence length = N

Then the formulation of this method is given by

$$x_1(n) = \{\underbrace{x(0), x(1), \ldots, x(L-1)}_{\text{'L' new data points}}, \underbrace{0, 0, 0, \ldots, 0}_{(M-1)\text{ zeros}}\}$$

$$x_2(n) = \{\underbrace{x(L), x(L+1), \ldots, x(2L-1)}_{\text{'L' new data points}}, \underbrace{0, 0, 0, \ldots, 0}_{(M-1)\text{ zeros}}\}$$

$$x_3(n) = \{\underbrace{x(2L), x(2L+1), \ldots, x(3L-1)}_{\text{'L' new data points}}, \underbrace{0, 0, 0, \ldots, 0}_{(M-1)\text{ zeros}}\}$$

and so on.

1. Find the o/p $y(n)$ of the filter whose impulse response $h(n) = \{1,1,1\}$ and i/p signal $x(n) = \{3,-1,0,1,3,2,0,1,2,1\}$ using overlap add method.

Sol: $N = L+M-1 = 10+3-1 = 12$

For reference, $N = 5$

$\Rightarrow \quad 5 = L+3-1$

$\therefore L = 3$

$x_1(n) = \{3,-1,0,0,0\}$

$x_2(n) = \{1,3,2,0,0\}$

$x_3(n) = \{0,1,2,0,0\}$

$x_4(n) = \{1,0,0,0,0\}$

$h(n) = \{1,1,1,0,0\}$

$$y_1(n) = \underset{h(n)}{\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}} \underset{x_1(n)}{\begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}} = \begin{bmatrix} 3 \\ 2 \\ 2 \\ -1 \\ 0 \end{bmatrix}$$

$\therefore y_1(n) = \{3,2,2,-1,0\}$

$$y_2(n) = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 6 \\ 5 \\ 2 \end{bmatrix}$$

$\therefore y_2(n) = \{1,4,6,5,2\}$

$$y_3(n) = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 3 \\ 2 \end{bmatrix}$$

$\therefore y_3(n) = \{0,1,3,3,2\}$

$$y_4(n) = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore y_4(n) = \{1,1,1,0,0\}$

```
3 2 2 -1 0
         1 4 6 5 2
                 0 1 3 3 2
                         1 1 1 0 0
-----------------------------------
3 2 2 0 4 6 5 3 3 4 2 1 0 0
```

$\therefore y(n) = \{3,2,2,0,4,6,5,3,3,4,2,1\}$

2. $x(n) = \{1,-1,2,-2,3,-3,4,-4\}$, $h(n) = \{-1,1\}$

Sol: $N = L+M-1 = 8+2-1 = 9$

For reference, $N=5$

$\Rightarrow 5 = L+2-1$

$\Rightarrow L = 4$

$x_1(n) = \{1,-1,2,-2,0\}$

$x_2(n) = \{3,-3,4,-4,0\}$

$h(n) = \{-1,1,0,0,0\}$

$$y_1(n) = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -3 \\ 4 \\ -2 \end{bmatrix}$$

$\therefore y_1(n) = \{-1,2,-3,4,-2\}$

$$y_2(n) = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 \\ +1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ -3 \\ 4 \\ -4 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \\ -7 \\ 8 \\ -4 \end{bmatrix}$$

$$\therefore y_2(n) = \{-3, 6, -7, 8, -4\}$$

-1	2	-3	4	-2				
				-3	6	-7	8	-4
-1	2	-3	4	-5	6	-7	8	-4

$$\therefore y(n) = \{-1, 2, -3, 4, -5, 6, -7, 8, -4\}$$